Problem 1 : Steffen and Greg

Since f is continuous on a bounded region we have that it must attain a maximum value $f(c)$, at c, on that range. Using Hölders inequality from page 139 (Rudin) and setting $f = f'$, $g = 1$, and $p = q = 2$ taking the integral from 0 to c we have

$$\Rightarrow \left| \int_0^c f' d\alpha \right| \leq \left\{ \int_0^c |f'|^2 d\alpha \right\}^{1/2} \left\{ \int_0^c |1|^1 d\alpha \right\}^{1/2} \tag{1}$$

$$\Rightarrow |f(c)|^2 \leq \int_0^c |f'|^2 d\alpha \times c \tag{2}$$

$$\Rightarrow |f(c)|^2 = ||f||_\infty^2 \leq \int_0^c |f'|^2 d\alpha \times c \leq \int_0^1 |f'(x)|^2 d\alpha \tag{3}$$

$$\Rightarrow ||f||_\infty^2 \leq \int_0^1 |f'(x)|^2 d\alpha \tag{4}$$

Problem 2 :

f_n being differentiable everywhere would imply that it is everywhere continuous. $f_n' \leq 2$ imples that f_n has bounded slope and it is easy to show that f_n is an equicontinuous family of functions (let $\delta = \frac{\varepsilon}{2}$). (In fact it can be shown that if any family of continuous functions has bounded slope that it is necessarily a equicontinuous family).

On any finite compact interval $[-A, A]$ we have that f is uniformly bounded $\Rightarrow$ pointwise bounded, since f is continuous on a compact region.

We appeal to theorem 7.25 to show that f_n must contain a uniformly convergent subsequence which must converge to $g(x)$. Along with $f_n(0) = 0$ implying pointwise convergence at at least one point we have that $\lim_{n\to\infty} f_n = g(x)$ unifomally.

Since f_n is continuous for any n we have that g(x) is also continuous by theorem 7.12. Taking $A \to \infty$ we have this convergence along the entire real line.

Problem 3 :

(a) Calculating $\hat{f}$ by the usual formula we have (integrating by parts) the following:

$$\hat{f'} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx = \frac{-1}{2\pi} \int_{-\pi}^{\pi} f(x) - ine^{-inx} dx = in \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f e^{-inx} dx \right) = in\hat{f} = c_n in$$

Which implies that $f'(x) \sim \sum_{n\in\mathbb{Z}} c_n ine^{inx}$

(b) We have that $c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{i0x} dx = \int_{-\pi}^{\pi} f(x) dx = 0$. And by Parseval's thm:

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx = 2\pi \sum_{-\infty}^{\infty} |c_n n|^2 \geq 2\pi \sum_{-\infty}^{\infty} |c_n|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

(c) We first assume that equality holds, and use Parsevals thm again (in (6))

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx = \int_{-\pi}^{\pi} |f(x)|^2 dx \tag{5}$$

$$\sum_{-\infty}^{\infty} |c_n n|^2 = \sum_{-\infty}^{\infty} |c_n|^2 \tag{6}$$

We know that equality can not hold for any $|n| > 2$ so (6) may be re-written as:

$$c_{-1}e^{-i(-1)x} + c_0 + c_1 e^{-inx} = c_{-1}e^{inx} + 0 + c_1 e^{-inx} \tag{7}$$

$$c_{-1}e^{inx} + c_1 e^{-inx} = ae^{inx} + be^{-inx} \tag{8}$$

To prove the other direction we set $f(x) = ae^{inx} + be^{-inx}$ and calculate $\hat{f}$ as usual.

$$\hat{f} = \frac{1}{2\pi}\int_{\pi}^{\pi} ae^{ix}e^{-inx}dx + \frac{1}{2\pi}\int_{\pi}^{\pi} be^{-ix}e^{-inx}dx \tag{9}$$

$$= \frac{1}{2\pi}\int_{\pi}^{\pi} ae^{ix(1-n)}dx + \frac{1}{2\pi}\int_{\pi}^{\pi} ae^{ix(-1-n)}dx \tag{10}$$

So we have that

$$\hat{f} = \begin{Bmatrix} a & n = 1 \\ b & n = -1 \\ 0 & otherwise \end{Bmatrix}$$

Which implies that

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx = 2\pi \sum_{-\infty}^{\infty} |c_n n|^2 = 2\pi(a^2 + b^2) = 2\pi \sum_{-\infty}^{\infty} |c_n|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx$$

as desired.

Problem 4 : Steffen

(b)

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n in|^2 \tag{11}$$

$$= \sum_{-\infty}^{\infty} c_n(\bar{c_n} n^2) \tag{12}$$

$$= \left(\sum_{-\infty}^{\infty} |c_n|^2\right)^{\frac{1}{2}} \left(\sum_{-\infty}^{\infty} |c_n n^2|^2\right)^{\frac{1}{2}} \tag{13}$$

$$\leq \left(\int |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int |f''(x)|^2 dx\right)^{\frac{1}{2}} \tag{14}$$

Line (11) is given by Problem 3 (a) where the transition from (13) to (14) is made by calculating $\hat{f''} = -c_n n^2$ again by an integration by parts and using the conclusions from (a).

Problem 5 :

(a) $\phi(x) = \frac{x^2+2}{4}$ will have max/min values at $x \in \{0, 1\}$ since $\phi(x)$ has only one critical point at 0. Since $\phi(0) = \frac{1}{2}$ and $\phi(1) = \frac{3}{4}$ we have that $\phi : [0, 1] \to [0, 1]$.

Since $|\phi'(x)| = \frac{3x^2}{4} < \frac{3}{4}$ we have that

$$\frac{|\phi(x) - \phi(y)|}{|x - y|} < \frac{3}{4} \Rightarrow |\phi(x) - \phi(y)| < \frac{3}{4}|x - y|$$

which implies ϕ is a contraction.

(b) We are guaranteed that ϕ has a fixed pt. N which means in the sequence $x_{n+1} = \phi(N) = N$. So $\{x_n\}$ limits to the fixed point of $\frac{x^3+2}{4}$ which may be established with a calculator through fixed point iteration. Namely, calculate $\phi(0)$ let this equal to x and calculate $\phi(x)$ repeat this process until your calculator starts repeating an answer. $N \approx 0.539171261$.

Problem 6 :

Lets assume that we have the root a such that $P(a) = 0$ the Problem dictates we must have $\phi(a) = a$.

$$0 = a^3 - 2a^2 - 9a + 4 \tag{15}$$

$$a = \frac{a^3 - 2a^2 + 4}{9} \tag{16}$$

$$\phi(a) = \frac{a^3 - 2a^2 + 4}{9} \tag{17}$$

Now prove ϕ is a contraction as in Problem 5.

Problem 7 :

We have that

$$u = e^y + x \qquad\qquad v = e^x - y \tag{18}$$

$$f_1 = u - e^y - x \qquad\qquad f_2 = v - e^x + y \tag{19}$$

So

$$A(x, y) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} -1 & -e^y \\ -e^x & 1 \end{bmatrix}$$

and

$$B(u, v) = \begin{bmatrix} \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial u & \partial f_2/\partial v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) We have $A(0,0) = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$ and since $A(0,0)$ is invertible, namely $A^{-1} = \frac{-1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ by Inverse Function Theorem we have that $f^{-1}(x,y)$ must exist about a neighborhood of $(0,0)$. So it is possible to express (x,y) as a differentiable function of (u,v) since this is exactly $f^{-1}(u,v) = (x,y)$.

(b) We know that

$$\begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix} = -A^{-1}(0,0)B(u(0,0),v(0,0)) = -A^{-1}(0,0)B(1,1)$$

So the desired derivatives may be extracted from

$$\begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Problem 8 :

Similarly in this Problem we have that

$$u = x^2 - y^2 \qquad v = 2xy \tag{20}$$

$$f_1 = u - x^2 + y^2 \qquad f_2 = v - 2xy \tag{21}$$

(a) To prove that the range of f is $\mathbb{R}$ we must show for any u and v there exists a corresponding x and y. Solving the equations in (20) for x and y by setting $y = \frac{v}{2x}$ and subbing into $x^2 = u + y^2$ and remembering that u and v are constants (you will be left with a quartic equation but just let $A = x^2$ and solve the corresponding quadratic equation). We can conclude that

$$x = \pm\sqrt{\frac{u \pm \sqrt{u^2+v^2}}{2}}$$

which yields only two real roots

$$x_1 = +\sqrt{\frac{u + \sqrt{u^2+v^2}}{2}} \qquad x_2 = -\sqrt{\frac{u + \sqrt{u^2+v^2}}{2}} \tag{22}$$

$$y_1 = \frac{v}{2x_1} \qquad y_2 = \frac{v}{2x_2} \tag{23}$$

So for any point (u,v) on the real plane we can find (x,y) such that $f(x,y) = (u,v)$ given by (20) and (21). This would imply that the range of f is $\mathbb{R}$. Furthermore it is clear that for any nonzero point (in the case of a zero the equations in (20) would generate the same point) we have two distinct points, (x_1,y_1) and (x_2,y_2) map to (u,v) by f.

(b) To show that the function is locally invertible at (1,1) we must simply show that the matrix $A(1,1)$ given in Problem 7 is invertible.

$$A(x,y) = \begin{bmatrix} -2x & 2y \\ -2y & -2x \end{bmatrix} \Rightarrow A(1,1) = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$$

which is invertible since $\det(A(1,1)) = 8$.

An explicit formula for the inverse is easy to find. Since if $f(x,y) = (u,v)$ an inverse would be given as $f^{-1}(u,v) = (x,y)$ which is exactly one of the equations given in (a). As we know that $f(1,1) = (0,2)$ we know that the inverse must carry $f^{-1}(0,2) = (1,1)$ which is only satisfied by

$$f^{-1}(u,v) = (x_1, y_1)$$

given above. (One should plug the numbers in to verify that this equation fails for $f^{-1}(u,v) = (x_2, y_2)$.)

Problem 9 :

Given more explicity in this question we have that

$$0 = wxyz \qquad w^4 + x^4 + y^4 + z^4 = 18 \tag{24}$$

$$f_1 = wxyz \qquad f_2 = w^4 + x^4 + y^4 + z^4 - 18 \tag{25}$$

(a)

$$A(x,y) = \begin{bmatrix} wyz & wxy \\ 4x^3 & 4y^3 \end{bmatrix}$$

at the point $(w,x,y,z) = (-1,0,1,2)$ becomes

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$$

Since the $\det(A) = -8$ we have that A is invertible, namely $A^{-1} = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$ which implies that it is possible to express (x,y) as a differentiable function of (w,z) near the given point.

(b) We calculate $B(w,z) = \begin{bmatrix} xyz & wxy \\ 4w^3 & 4z^3 \end{bmatrix} \Rightarrow B(-1,2) = \begin{bmatrix} 0 & 0 \\ -4 & 32 \end{bmatrix}$ so we can calculate the partial derivatives by

$$\begin{bmatrix} \partial x/\partial w & \partial x/\partial z \\ \partial y/\partial w & \partial y/\partial z \end{bmatrix} = -A^{-1}(0,1)B(-1,2) = \begin{bmatrix} 0 & 0 \\ 1 & -8 \end{bmatrix}$$

So $\frac{\partial x}{\partial w}(-1,2) = 0$ and $\frac{\partial x}{\partial z}(-1,2) = 0$.

(c) To calculate the partial derivatives explicitly we have to proceed as we did in Problem 7 to try to come up w/ explicit formulas for x and y. We have $wxyz = 0$ implies that $x = 0$ or $y = 0$. We take $x = 0$ since it satisfies $\frac{\partial x}{\partial w}(-1,2) = \frac{\partial x}{\partial w}(-1,2) = 0$. So solving for y from (24) we get

$$y = \sqrt[4]{18 - z^4 - w^4}$$

and conclude that

$$\frac{\partial y}{\partial w}(w,z) = \frac{-4w^3}{3(18 - w^4 - z^4)^{3/4}} \qquad \frac{\partial y}{\partial z}(w,z) = \frac{-4z^3}{3(18 - w^4 - z^4)^{3/4}} \tag{26}$$

$$\frac{\partial y}{\partial w}(-1,2) = 1 \qquad \frac{\partial y}{\partial w}(-1,2) = -8 \tag{27}$$

which verifies (b).