§ POWER SERIES

Consider the "power series"

$$S(x) = 1 + x^2 + x^3 + \cdots = \sum_{K=0}^{\infty} x^K .$$

This power series can be evaluated:

$$S\left(\tfrac{1}{2}\right) = 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^3 + \cdots$$

$$= \frac{1}{1 - \frac{1}{2}} = 2 .$$

QUESTION: For what x does $S(x)$ converge?

ANSWER: Notice $S(x)$ is a geometric series which converges for $|x| < 1$.

Prop$^{\underline{n}}$ $|x| < 1 \Rightarrow 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$

"Proof" Graphically using DESMOS

Def$^{\underline{n}}$ A power series "about" or "centered" at $x = 0$

$$\sum_{n=0}^{\infty} C_n x^n = C_0 x^0 + C_1 x^1 + C_2 x^2 + \cdots$$

"about" or "centered" at $x = a$

$$\sum_{n=0}^{\infty} C_0 (x-a)^n = C_0 (x-a)^0 + C_1 (x-a)^1 + \cdots$$

cont...

... where a is called "the center" and $c_0, c_1, c_2 \ldots \in \mathbb{R}$ are "coefficients"

<u>EXAMPLE</u> $\sum x^n = 1 + x + x^2 + x^3 + \cdots$
$= 1 + (x-0) + (x-0)^2 + \cdots$

has <u>center</u> $a=0$ and converges for $|x|<1$.

<u>EXAMPLE</u> $S(x) = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{2-x}{2}\right)^n$$ ... a geometric series w/ $r = \left(\frac{2-x}{2}\right)$

provided $\left|\frac{2-x}{2}\right| < 1 \Rightarrow -1 < \frac{2-x}{2} < 1.$

$\Rightarrow -2 < 2-x < 2 \Rightarrow -4 < -x < 0$
$\Rightarrow 0 < x < 4 \Rightarrow x \in (0,4)$

We have $S(x) = \dfrac{1}{1-\left(\frac{2-x}{2}\right)} = \dfrac{2}{x}$

EXAMPLE For what values of x does

$$S(x) = \sum_{K=0}^{\infty} K!\, x^K \text{ converge?}$$

Ratio test: $\dfrac{a_{K+1}}{a_K} = \dfrac{(K+1)K!\, x \cdot x^K}{K!\, x^K} = (K+1)x$

$\lim\limits_{K\to\infty} \dfrac{a_{K+1}}{a_K} = \lim\limits_{K\to\infty} (K+1)x$ is divergent when $x \neq 0$.

Thm Convergence Thm

Let $S(x) = \sum a_n x^n$ be a power series

if $S(c)$ converges for $c \neq 0$ then

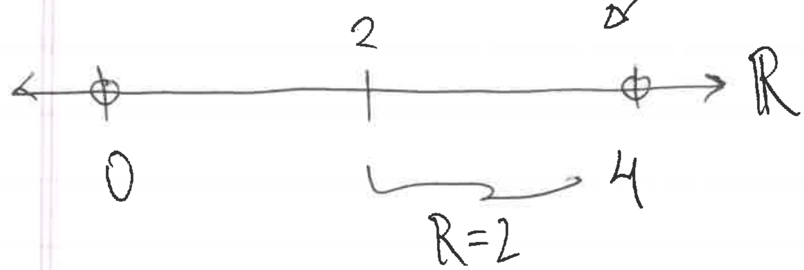
$S(x)$ abs converges for $|x| < |c|$.

Also $S(d)$ divergent $\Rightarrow$ $S(x)$ div for $|x| > |d|$

Radius of Convergence

For $\sum \left(\frac{2-x}{2}\right)^n$ we have

convergence inside int.



Radius of convergence is $R=2$.

Interval of convergence is $(0,4)$

Center $c=2$.

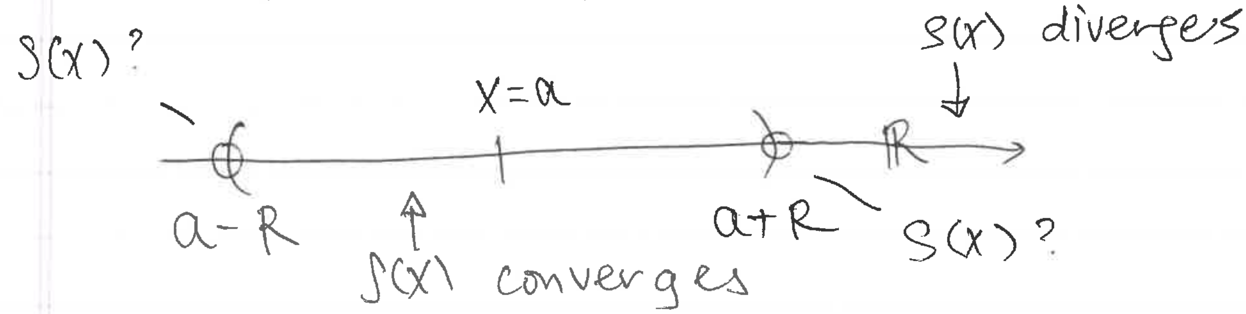
Thm Let $\sum_{k=0}^{\infty} c_k (x-a)^k = S(x)$ either (i.e. exactly one)

① There is $R \in \mathbb{R}^{>0}$ such that
- $S(x)$ diverges $|x-a|>R$
- $S(x)$ converges for $|x-a|<R$
- con/div at endpoints $x=a-R,\ a-R$

(i.e. "inconclusive")

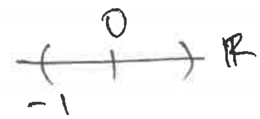
② $S(x) < \infty$ for ~~$x=a$ only~~ every x.
(i.e. radius $R=\infty$).

3. $S(x) < \infty$ for $x = a$ only (Radius $R = 0$)

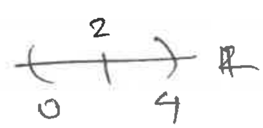


EXAMPLE

- $\sum x^k \quad R = 1,\ C = 0$



- $\sum \left(-\frac{1}{2}\right)^n (x-2)^n \quad R = 2,\ C = 2$



- $\sum k!\,x^k \quad R = 0,\ C = 0$

Thm If $\sum C_n (x-a)^n$ converges for $x \in (a-R, a+R) : R > 0$ then

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

is a function over $x \in (a-R, a+R)$ whose derivatives

$$f'(x) = \sum_{n=1}^{\infty} n \cdot C_n (x-a)^{n-1} \qquad f''(x) = \sum_{n=2}^{\infty} n(n-1) C_n (x-a)^{n-2}$$

... also converge for $x \in (a-R, a+R)$.

<u>EXAMPLE</u>: $S(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{K=0}^{\infty} x^K$

$\Rightarrow S'(x) = 0 + 1 + 2x + 3x^2 + \cdots = \sum_{K=1}^{\infty} K \cdot x^{(K-1)}$

$\Rightarrow S''(x) = 0 + 2 + 2 \cdot 3x + \cdots = \sum_{K=2}^{\infty} K \cdot (K-1) x^{K-2}$

Notice the index $K = 0, 1, 2$ because we are losing constant terms as we differentiate.

<u>WARNING</u> This is <u>not</u> generally applicable outside power series.

$$S(x) = \sum_{n=1}^{\infty} \frac{\sin(n! x)}{n^2} \quad \text{converges for } x \in \mathbb{R}$$

but $S'(x) = \sum_{n=1}^{\infty} \frac{\cos(n! x) n!}{n^2}$ diverges for $x \in \mathbb{R}$.

<u>Thm</u> Suppose $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$ <u>converges</u> for $x \in (a-R, a+R) : R > 0$. <u>then</u>

$F(x) = \sum_{n=0}^{\infty} \frac{C_n (x-a)^{n+1}}{n+1}$ is convergent for $x \in (a-R, a+R)$

and $\int f(x)\,dx = \sum_{n=0}^{\infty} C_n \frac{(x-a)^{n+1}}{n+1} + C = F(x) + C \cdot n$ as $n \to \infty$

How can we numerically approximate ln?

<u>EXAMPLE</u> Notice $1-t+t^2-t^3+\cdots$

$$= \sum (-1)^n t^n = \sum (-t)^n = \frac{1}{1+t}$$

converges for $t \in (-1,1)$. To approx $\ln(x)$ recognize

~~Thus~~ $$\int_0^x \frac{1}{1+t}\,dt = \int_0^x 1-t+t^2-t^2+\cdots\,dt$$

$$= t - t^2/2 + t^3/3 - t^4/4 + \cdots \Big|_0^x$$

$$= x - x^2/2 + x^3/3 - x^4/4 + \cdots$$

But recall: $$\int_0^x \frac{1}{1+t}\,dt = \ln(1+t)\Big|_0^x = \ln(1+x)$$

$$\Rightarrow \ln(1+x) = x - x^2/ + x^3/3 - x^4/4 + \cdots$$

$$\Rightarrow \ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

An approximate for $\ln(x)$ that a calculator could perform.

EXAMPLE Consider $S(x) = \sum_{K=0}^{\infty} \left(\frac{x}{5}\right)^K$

Find interval of convergence for $S(x)$

$$a_K = \left(\frac{x}{5}\right)^K \Rightarrow \frac{a_{K+1}}{a_K} = \frac{x \cdot x^K}{5 \cdot 5^K} \cdot \frac{5^K}{x^K} = \frac{x}{5}$$

and $\lim_{K \to \infty} \left|\frac{x}{5}\right| < 1 \Leftrightarrow |x| < 5$

By ratio test $S(x)$ abs-converges on $(-5, 5)$

- Find a formula/closed form for $S(x)$

$$x \in (-5,5) \rightarrow S(x) = \frac{1}{1 - x/5} = \frac{5}{5-x}$$

- Find $S'(x)$: $S'(x) = \sum_{K=0}^{\infty} K \cdot \left(\frac{x}{5}\right)^{K-1} \cdot \frac{1}{5}$

- Find closed-form for $S'(x)$:

$$S(x) = \frac{5}{5-x} \Rightarrow S'(x) = \frac{5}{(5-x)^2}$$

- Find $\sum_{n=1}^{\infty} \frac{n}{5^n} = \sum_{n=0}^{\infty} \frac{n}{5^n} = S'(1) = \frac{5}{(5-1)^2} = \frac{5}{16}.$

§ Taylor Series

We use Taylor Series to give polynomial approximations of ~~ab~~ arbitrary diff'able functions

For instance, suppose we want to approximate $f(x)$ near $x=a$. w/ a power series $p(x) = \sum \cancel{a}\, c_K (x-a)^K$

We would need, near $x=a$, that

$f'(a) = p'(a)$ same slope

$f''(a) = p''(a)$ same concavity

$f'''(a) = p''(a)$ same inflection

$\vdots$

$f^{(K)}(a) = p^{(K)}(a)$ higher order sameness

$\cancel{p(x) = a_0 + a_1 x}$

$$P(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$$

$$P'(x) = C_1 + C_2 \cdot 2 \cdot (x-a) + C_3 \cdot 3(x-a)^2 + \cdots$$

$$P''(x) = C_2 \cdot 2 + C_3 \cdot 3 \cdot 2 \cdot (x-a) + C_4 \cdot 4 \cdot 3 \cdot (x-a)^2 + \cdots$$

$$P'''(x) = C_3 \cdot 3 \cdot 2 \cdot 1 + C_4 \cdot 4 \cdot 3 \cdot 2 \cdot (x-a) + \cdots$$

$$\vdots$$

$$P^{(K)}(x) = C_K K! + \cdots$$

Also $P'(a) = C_1$

$$P''(a) = 2!\, C_2$$

$$P'''(a) = 3!\, C_3$$

$$P^{(K)}(a) = K!\, C_K$$

So, at "$x=a$" a function $f(x)$:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a) + \cdots$$

<u>Defn</u> Taylor Series

For a function $f(x)$, its <u>Taylor Series</u> at $x=a$ is:

$$P(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

<u>Defn</u> Taylor Polynomial

The partial sum

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the <u>Taylor Polynomial</u> of <u>order n</u>.

<u>Defn</u> Maclaurin Series

When $a=0$ the Taylor Series is sometimes referred to as a <u>Maclaurin Series</u>.

EXAMPLE Find the Taylor Series for $g(x) = \ln(x)$ at $x = 1$.

$$g^{(0)}_{(x)} = \ln(x) \qquad g^{(3)}_{(x)} = \frac{2\cdot 1}{x^3}$$

$$g^{(1)}_{(x)} = \frac{1}{x} \qquad g^{(4)}_{(x)} = \frac{-3\cdot 2\cdot 1}{x^4}$$

$$\vdots$$

$$g^{(2)}_{(x)} = \frac{-1}{x^2} \qquad g^{(K)}_{(x)} = \frac{(-1)^{K+1}(K-1)!}{x^K} \; ; \quad K > 0$$

So
$$g^{(K)}_{(1)} = \begin{cases} (-1)^{K+1}(K-1)! & K > 0 \\ \ln 1 = 0 & K = 0 \end{cases}$$

Thus
$$P(x) = \sum_{K=0}^{\infty} \frac{g^{(K)}_{(1)}}{K!}(x-1)^K = \sum_{K=1}^{\infty} \frac{(-1)^{K+1}(K-1)!}{K!}(x-1)^K$$

$$\boxed{= \sum_{K=1}^{\infty} \frac{(-1)^{K+1}}{K}(x-1)^K}$$

Also,

$P_0 = 0$, $P_1 = (x-1)$, $P_2 = (x-1) - \frac{1}{2}(x-1)^2$

$P_3 = (x-1) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-1)^3$,

$P_4 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$