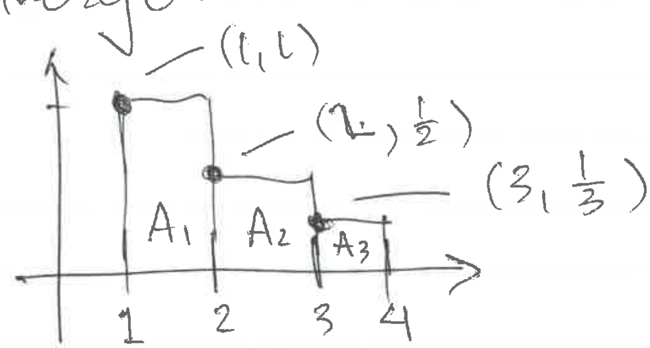
$\int$ Integral Test

Consider the so-called "Harmonic Series".

$$\sum_{K=1}^{\infty} \frac{1}{K} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Does it converge?

Notice:



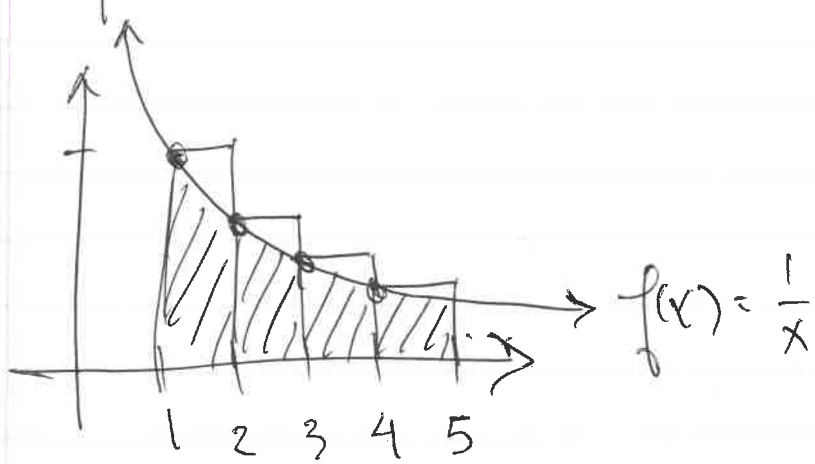
$A_1 = 1 \times 1$

$A_2 = 1 \times \frac{1}{2}$

$A_3 = 1 \times \frac{1}{3}$

$\vdots$

We can represent a series ~~using~~ by the area of a bar plot.



Area under $\frac{1}{x}$ is less than the series' sum.

Integral is <u>less than</u> $A_1 + A_2 + A_3 + \cdots = \sum_{K=1}^{\infty} \frac{1}{K}$:

$$\int_1^{\infty} \frac{1}{x}\,dx \leq \sum_{K=1}^{\infty} \frac{1}{K}$$

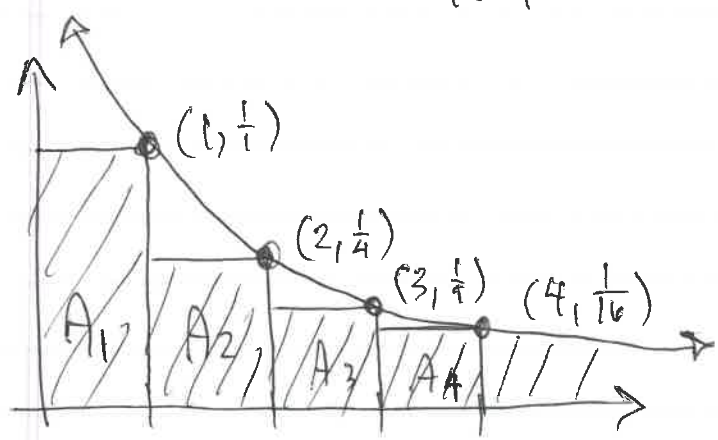
where $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{N \to \infty} \ln|x| \Big|_1^N$

$= \lim_{N \to \infty} \ln|N| - \ln|1| = \lim_{N \to \infty} \ln|N| \text{ "="} \infty$.

Divergent.

Thus $\sum_{K=1}^{\infty} \frac{1}{K}$ diverges as well.

<u>QUESTION</u>: Does $\sum_{K=1}^{\infty} \frac{1}{K^2}$ converge or diverge?



$$A_1 + A_2 + A_3 + A_4 + \cdots = \sum_{K=1}^{\infty} \frac{1}{K^2} \underset{\substack{\uparrow \\ \text{strict}}}{<} \int_1^{\infty} \frac{1}{x^2}\,dx = 2.$$

where $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{N\to\infty} \left. -\frac{1}{x} \right|_1^{\infty} = \lim_{N\to\infty} -\frac{1}{N} + 1 = 0 + 1 = 1.$

so $\sum_{K=1}^{\infty} \frac{1}{K^2} < 2 \Rightarrow$ sum diverges

<u>Thm</u> Integral Test

- let $\{a_n\}$ be a real sequence.
- suppose $f: \mathbb{R} \to \mathbb{R}$ and $f(n) = a_n$ w/ $f(x)$ is cont, pos., dec $\forall x \geq N$

(i.e. "eventually") . . .

Integral Test cont...

Then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$

diverge/converge together.

EXAMPLE: When does the "p-series" converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

As $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{N \to \infty} \left. \frac{x^{-p+1}}{(-p+1)} \right|_1^N = \lim_{N \to \infty} \left. \frac{1}{(1-p)x^{p-1}} \right|_1^N$

$= \frac{1}{1-p}\left[\lim_{N \to \infty} \frac{1}{N^{p-1}} - \frac{1}{1^{p-1}} \right]$ which is

convergent when $\lim_{N \to \infty} \frac{1}{N^{p-1}} < \infty$ (i.e. convergent)

$\iff p-1 > 0 \iff p > 1.$

Propⁿ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ "the p-series" converges when $p>1$. and diverges otherwise.

~~ERROR~~ EXAMPLE Does $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ converge?

(It is not a p-series).

Notice $f(x) = \frac{1}{x^2+1}$ has

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \arctan x \Big|_1^b$$

$$= \lim_{b\to\infty} \arctan b - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

By integral test sum converges.

$\frac{\pi}{4}$ is not the value of the sum.

EXERCISE Determine conv/div?

- $\sum_{n=1}^{\infty} n e^{-n^2}$
- $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

§ Error Estimation

Suppose $\sum_{k=0}^{\infty} a_k = S$. We need ways to answer questions like:

- How many terms of this series would we need to sum in order to be within 0.0001 of S?

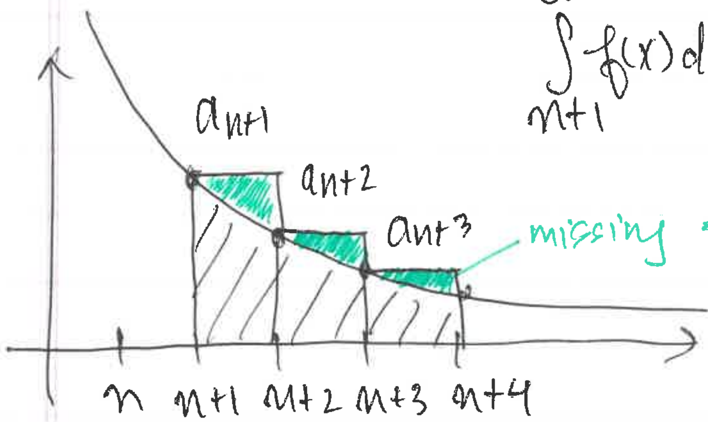
Denote the "remainder"

$$R_n = \underset{\substack{\uparrow \\ \text{actual/} \\ \text{exact sum}}}{S} - \underset{\substack{\uparrow \\ \text{partial/} \\ \text{approx sum.}}}{s_n} = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

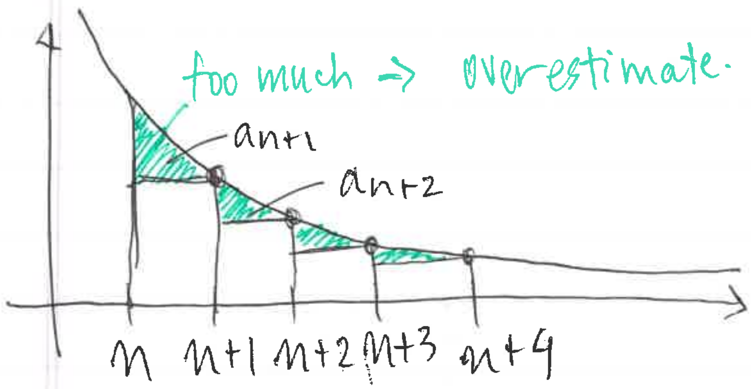
The remainder is all the terms we didn't bother adding.

Suppose $f(x) = a_n$ for $x \in \mathbb{N}^{\geq 1}$

and consider $R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots$



$$\int_{n+1}^{\infty} f(x)\,dx \leq a_{n+1} + a_{n+2} + \cdots = R_n$$



$$\int_{n}^{\infty} f(x)\,dx \geq a_{n+1} + a_{n+2} + \cdots = R_n$$

Prop$^{\underline{n}}$ Bounds for remainder in Integral Test

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

OR EQUIV.

$$S \in \left[S_n + \int_{n+1}^{\infty} f(x)\,dx,\ S_n + \int_{n}^{\infty} f(x)\,dx \right]$$

EXERCISE Let $S = \sum_{K=1}^{\infty} \frac{1}{K^2}$.

Find a value, $\hat{s}$, such that

$$|S - \hat{s}| \le \frac{1}{100}.$$

We're essentially trying to find a "good" estimation of S.

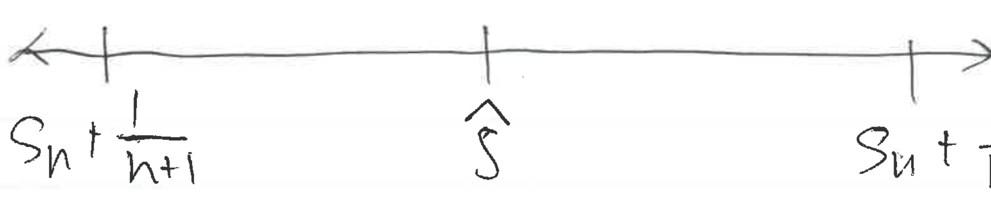
We know $S \in \left[S_n + \int_{n+1}^{\infty} \frac{1}{x^2}\,dx \,,\, S_n + \int_{n}^{\infty} \frac{1}{x^2}\,dx \right]$

where $\int_{n+1}^{\infty} \frac{1}{x^2}\,dx = \lim_{b \to \infty} \left. \frac{-1}{x} \right|_{n+1}^{b} = \lim_{b \to \infty} -\frac{1}{b} + \frac{1}{n+1} = \frac{1}{n+1}$

$$\int_{n}^{\infty} \frac{1}{x^2}\,dx = \cdots = \frac{1}{n}$$

So $S \in \left[S_n + \frac{1}{n+1} \,,\, S_n + \frac{1}{n} \right]$, i.e.

We know S — the exact sum — falls somewhere in the interval. We guess middle



$\cancel{\Rightarrow \hat{S} = S_n + \frac{\frac{1}{n} + \frac{1}{n+1}}{2} = S_n + \frac{2n+1}{2n(n+1)}}$

$$\Rightarrow \hat{S} = \frac{\left(S_n + \frac{1}{n}\right) + \left(S_n + \frac{1}{n+1}\right)}{2}$$ the average of the two points.

$$= \frac{2S_n + \frac{2n+1}{n(n+1)}}{2} = S_n + \frac{2n+1}{2n(n+1)}$$

The distance between $\hat{S}$ and S cannot exceed half the length of the interval.

Thus $$|S - \hat{S}| \leq \frac{\left(S_n + \frac{1}{n}\right) - \left(S_n + \frac{1}{n+1}\right)}{2}$$

$$\leq \frac{1}{2n(n+1)}$$

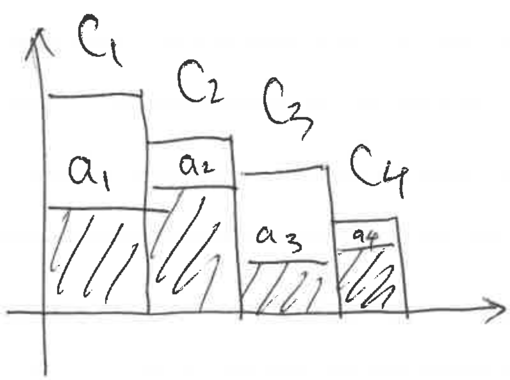
when $n = 7$ $\leq \frac{1}{2 \cdot 7 \cdot 8} \leq \frac{1}{100}$.

This means $S_7 + \frac{2 \cdot 7 + 1}{2 \cdot 7 \cdot 8} \approx 1.64$ is to w/in 0.01 of S.

$S_{10^{10}} \approx 1.64572562$ via my computer.

§ Comparison Test

Intuition:



Thm Comparison Test

Let $\sum a_n$ & $\sum c_n$, $\sum d_n$ be series w/ $a_n, c_n, d_n \geq 0$. Suppose $d_n \leq a_n \leq c_n$ eventually.

(A) $\sum c_n$ converges $\Rightarrow$ $\sum a_n$ converges

(B) $\sum d_n$ diverges $\Rightarrow$ $\sum a_n$ diverge

EXAMPLE: $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1} = \sum \frac{1}{n-\frac{1}{5}} \ngeq \sum \frac{1}{n}$

By CT p B. $\sum \frac{5}{5n-1}$ converges.

Thm Limit Comparison Test (LCT)

Suppose $a_n > 0$, $b_n > 0$ eventually. Then

(A) $0 < \lim_{b \to \infty} \frac{a_n}{b_n} < \infty \Rightarrow \sum a_n$ and $\sum b_n$ con/div together.

(B) $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n < \infty$

$\Rightarrow \sum a_n < \infty$

(C) $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges

$\Rightarrow \sum a_n$ diverges.

EXAMPLE: Does $3/4 + 5/9 + 7/16 + 9/25 + \cdots$ con or div?

$a_n = \frac{3+2n}{(2+n)^2} \Rightarrow \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2n}{n^2} = \lim_{n \to \infty} \frac{2}{n}$

Notice if $b_n = \frac{1}{n} \Rightarrow \sum b_n$ divergent and

$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2/n}{1/n} = 2$

by LCT p B $\sum a_n = 3/4 + 5/9 + \cdots$ diverges.

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{(n^5+1)^{\frac{1}{2}}}$ con or div ?

let $a_n = \dfrac{n+1}{(n^5+1)^{\frac{1}{2}}} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n}{n^{5/2}} = \lim_{n\to\infty} \dfrac{1}{n^{3/2}}$

$= 0.$

$b_n = \dfrac{1}{n^{3/2}} \Rightarrow \lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{1/n^{3/2}}{1/n^{3/2}} = 1$

LCTpA $\Rightarrow \sum a_n, \sum b_n$ con/div together.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges because it is p-series w/ $p>1$.

EXAMPLE $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ let $a_n = \frac{2^n+3^n}{3^n+4^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n+3^n}{3^n+4^n} = \lim_{n\to\infty} \frac{3^n}{4^n} = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n$$

$$= 0.$$

$$b_n = \left(\frac{3}{4}\right)^n \Rightarrow \lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{(3/4)^n}{(3/4)^n} = 1$$

Notice $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$

$$= \frac{3}{4} \cdot \frac{1}{1-\frac{3}{4}} = 3.$$

By LCTpA $\sum a_n$ converges

EXAMPLE $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ con or div?

Intuition: $\sin x \approx x$ for x small.

Notice "Eventually" $\frac{1}{n}$ is small.

Thereby: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) = \sum_{n=1}^{N-1} \sin\left(\frac{1}{n}\right) + \sum_{n=N}^{\infty} \frac{1}{n}$

$\sin x = x$

and $\sum \frac{1}{n}$ is divergent.

Let $a_n = \sin\left(\frac{1}{n}\right) \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \lim_{\theta\to 0} \sin\theta$

$b_n = \frac{1}{n} \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = \lim_{\theta\to 0} \theta$ w/ $\theta = \frac{1}{n}$.

$\rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{1/n} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

LCT p A $\Rightarrow \sum a_n$ is divergent because $\sum b_n = \sum \frac{1}{n}$ is div.