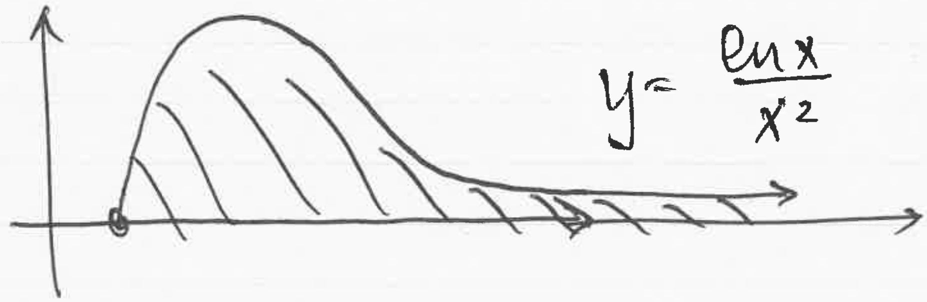
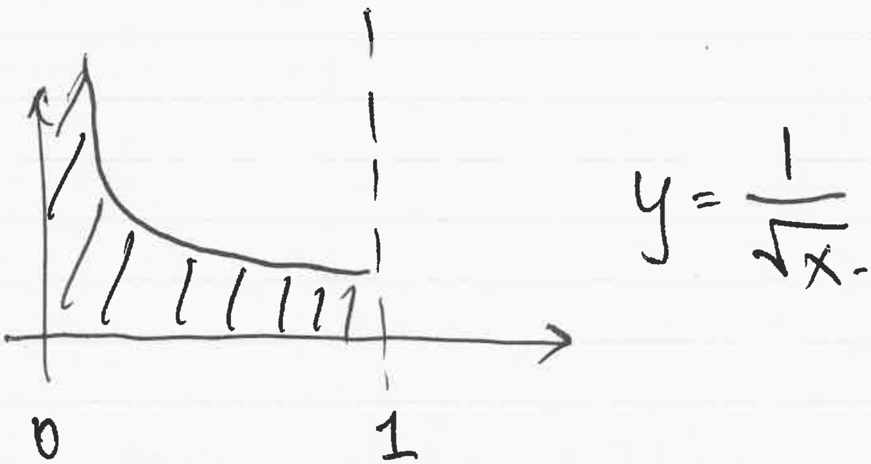
$\int$ Improper Integrals

Motivation: What is the bounded area of





Is the area finite / convergent / $< \infty$?

Is the area infinite / divergent ?

We sometimes say $x < \infty$ instead of "x is convergent."

EXAMPLE: $y = \dfrac{\ln x}{x^2}$

$$\text{"BA"} = \lim_{n\to\infty} \int_1^n \frac{\ln x}{x^2}\,dx$$

$u = \ln x \qquad v = -\frac{1}{x}$

$du = \frac{1}{x}dx \qquad dv = \frac{1}{x^2}dx$

$$= \lim_{n\to\infty} \left[\frac{-\ln x}{x} - \int -\frac{1}{x}\frac{1}{x}\,dx \right]_1^n$$

$$= \lim_{n\to\infty} \frac{-1}{x} - \frac{\ln x}{x} \Bigg|_1^n$$

$$= \lim_{n\to\infty} \left(\frac{-1}{n} - \frac{\ln n}{n}\right) - \left(\frac{-1}{1} - \frac{\ln 1}{1}\right)$$

$$= 0 - \lim_{n\to\infty} \frac{\ln n}{n} + 1 + 0$$

LH

$$= 1 - \lim_{n\to\infty} \frac{1/n}{1} = 1 - 0 = 1.$$

Notation

$$\int_{a}^{\infty} f(x)\,dx := \lim_{b\to\infty} \int_{a}^{b} f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

EXAMPLE $y = \frac{1}{\sqrt{x}}$ $x \in [0,1]$ has an asymptote at $x=1$.

$$\text{"BA"} = \lim_{a \to 0^-} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a \to 0^-} 2\sqrt{x}\,\Big|_a^1$$

$$= \lim_{a \to 0^-} \left(2\sqrt{1} - 2\sqrt{a}\right) = 2 - 0 = 2.$$

EXAMPLE

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \arctan x\,\Big|_a^0 + \lim_{b \to \infty} \arctan x\,\Big|_0^b$$

$$= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right) = \pi.$$

EXAMPLE $\int_0^1 \frac{1}{1-x}\,dx$ has an asymptote

$$= \lim_{b\to 1^-} \int_0^b \frac{1}{1-x} = \lim_{b\to 1^-} \ln|1-x| \Big|_0^b$$

$$= \lim_{b\to 1^-} -\ln|1-b| + \ln|1-0| = +\infty$$

You can integrate "through" asymptotes.

EXAMPLE $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$

$$= \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^{2/3}}\,dx + \lim_{a\to 1^+} \int_a^3 \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \cdots = 3 + 3\sqrt[3]{2}.$$

$\S$ Convergence/Divergence

Prop$^{\underline{n}}$ Direct Comparison Test.

When $0 \leq f(x) \leq g(x)$ for $x \in [a, \infty)$

$$\int_a^\infty g(x)\,dx \text{ converges} \Rightarrow \int_a^\infty f(x)\,dx \text{ converges}$$

$$\int_a^\infty f(x)\,dx \text{ diverges} \Rightarrow \int_a^\infty g(x)\,dx \text{ diverges}$$

Proof: Notice $0 \leq f(x) \leq g(x) \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

thereby $\int_a^\infty g(x)\,dx < \infty \Rightarrow \int_a^b f(x)\,dx \leq \int_a^\infty g(x)\,dx < \infty$

$\Rightarrow \int_a^\infty f(x)\,dx < \infty$ – convergent.

The "contrapositive" gives

$\int_a^\infty f(x)\,dx$ <u>not</u> convergent $\Rightarrow \int_a^\infty g(x)\,dx$ <u>not</u> convergent.

EXAMPLE Does $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converge?

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left.\frac{-1}{x}\right|_1^b = 0+1 = 1$$

$$\Rightarrow \int_1^\infty \frac{1}{x^2}\,dx < \infty \quad \text{Convergent}$$

But $0 \le \sin^2 x \le 1 \Rightarrow 0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$

thereby $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx < \infty$

$\Rightarrow \int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

Thm Limit Comparison Test

Let $f(x), g(x) \ge 0$ for $x \in [a, \infty)$

$0 < \lim_{x\to\infty} f(x)/g(x) = L < \infty$

$\Rightarrow \int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge

EXAMPLE: Does $\int_1^\infty \frac{1}{1+x^2}\,dx$ converge?

Notice: $\int_1^\infty \frac{1}{x^2}\,dx < \infty$ is convergent

and $\lim_{x\to\infty} \frac{1/x^2}{1/(1+x^2)} = \frac{1+x^2}{x^2} = 1$

Thus, by LCT, $\int_1^\infty \frac{1}{1+x^2}\,dx$ is convergent.