§AVERAGES

Let $X = \{x_1, \dots, x_N\}$

The $AVE(X) = \dfrac{X_1 + \cdots + X_N}{N} = \dfrac{1}{N}\sum_{K=1}^{N} X_K = \sum_{K=1}^{N} X_K \cdot \dfrac{1}{N}$

This looks pretty close to a Riemann Sum

Consider

$$\frac{1}{B-A}\int_A^B f(x)\,dx \overset{LEE}{=} \frac{1}{B-A}\lim_{N\to\infty}\sum_{K=1}^{N} f(A+K\Delta X)\,\Delta X$$

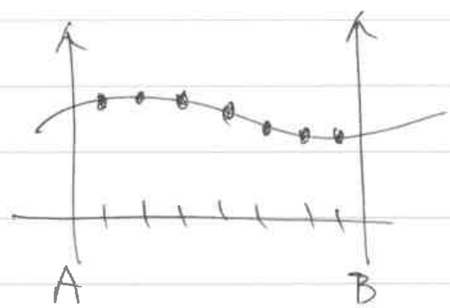
w/ $\Delta X = \dfrac{B-A}{N}$

For any $N<\infty$ (i.e. finite N) we have

$$\frac{1}{B-A}\sum_{K=1}^{N} f(A+\Delta X)\frac{B-A}{N} = \frac{1}{N}\sum_{K=1}^{N} f(A+K\Delta X)$$

$$= AVE\big(f(A+\Delta X), f(A+2\Delta X), \dots, f(B)\big)$$

which we can see as the average "height" of the points...



In the limit we have:

$$\frac{1}{B-A}\int_A^B f(x)\,dx = AVE(f(x^*) : x^* \in [A,B])$$

Propⁿ The Average Value of an integrable $f(x)$ over $[a,b]$ is

$$AVE(f(x) : x \in [a,b]) = \int_A^B f(x)\,dx \cdot \frac{1}{B-A}$$

EXAMPLE What is the average ~~...~~ height of $\cos\theta$ over $[0, 2\pi]$?

$$AVE(\cos\theta : \theta \in [0, 2\pi]) = \frac{1}{2\pi - 0}\int_0^{2\pi} \cos(\theta)\,d\theta$$

$$= \frac{\sin\theta}{2\pi}\Big|_0^{2\pi} = 0 - 0 = 0.$$

... over $[0, \pi/2]$?

$$AVE(\cos\theta : \theta \in [0, \tfrac{\pi}{2}]) = \frac{1}{\frac{\pi}{2}}\int_0^{\pi/2} \cos\theta\,d\theta = \frac{2}{\pi}\sin\theta\Big|_0^{\frac{\pi}{2}} = \frac{2}{\pi}.$$

EXERCISE Find average height of

- $f(x) = x^2$ over $[-2, 2]$
- $\sqrt{1-x^2}$ over $[-1, 1]$.