∫ INTEGRATION BY TABLE LOOKUP

Given a rule for taking an integral — you are expected to be able to use it.

Rule: $\int \frac{1}{(a^2 - x^2)^{\frac{1}{2}}}\, dx = \arcsin\left(\frac{x}{a}\right) + C$ (1)

...we will prove this later.

EXAMPLE: $\int \frac{1}{(8x - x^2)^{\frac{1}{2}}}\, dx = -\int \frac{1}{(4^2 - u^2)^{\frac{1}{2}}}\, du = (*)$

let $u = 4 - x \rightarrow du = -dx$

$(*) = \underbrace{-\arcsin\left(\frac{u}{4}\right) + C}_{\text{by (1)}} = C - \arcsin\left(\frac{4-x}{4}\right)$

$\boxtimes$

Rule: $\int \sec^2 x\, dx = \tan x + C$ (2)

Rule: $\int \sec x \tan x\, dX = \sec X + C$ (3)

EXAMPLE: $\int_0^{\pi/4} \frac{1}{1-\sin x}\,dx = \int_0^{\pi/4} \frac{1}{1-\sin x}\,\frac{1+\sin x}{1+\sin x}\,dx$

$$= \int_0^{\pi/4} \frac{1+\sin x}{1-\sin^2 x}\,dx = \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx = \int_0^{\pi/4} \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int_0^{\pi/4} \sec^2 x + \sec x \tan x\,dx = \Big[\tan x + \sec x\Big]_0^{\pi/4}$$

$$= (1+\sqrt{2}) - (0+1) = \sqrt{2}. \quad \boxtimes$$

There are *many* integration rules — in fact there are books full of just rules

§ INTEGRATION BY PARTS

Basically reverse product rule.

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'g + fg'\,dx$$

$$\Rightarrow \int f g'\,dx = \int \frac{d}{dx}(f\cdot g)\,dx - \int f'g\,dx = (*)$$

Letting $u = f(x) \Rightarrow du = f'(x)\,dx$

$v = g(x) \Rightarrow dv = g'(x)\,dx$

$$(*) = \boxed{\int u\,dv = uv - \int v\,du}$$

EXAMPLE: $\int \overbrace{x}^{u}\ \overbrace{\cos x\,dx}^{dv} = x\sin x - \int \sin x\,dx = (*)$

$u = x$ $\quad$ $v = \sin x$

$du = dx$ $\quad$ $dv = \cos x\,dx$

$(*) = x\sin x + \cos x + c$

Confirm: $(x\sin x + \cos x + c)' = \sin x + x\cos x - \sin x + c$
$= x\cos x$. ✓ $\boxtimes$

Remember about "silent ones" — that is to say $f(x)$ is always the product $1 \cdot f(x)$.

EXAMPLE: $\int \ln(x)\,dx = (*)$

$u = \ln(x)$ $\quad\quad$ $v = x$

$du = \frac{1}{x}dx$ $\quad\quad$ $dv = 1 \cdot dx$

$(*) = x\ln(x) - \int x \cdot \frac{1}{x}dx = x\ln(x) - \int 1\,dx$

$= x\ln(x) - x + C$

CHECK: $(x\ln x - x + C)' = \ln x + x \cdot \frac{1}{x} - 1 = \ln x$ ✓

$\boxtimes$

$\int$ Recursive / Circular IBP

EXAMPLE: $\int e^x \cos x \, dx$ (WORKSHEET)

$u = e^x$ $\quad\quad$ $v = \sin x$

$du = e^x \, dx$ $\quad\quad$ $dv = \cos x \, dx$

$(*) = \int e^x \cos x \, dx = e^x \sin x - \underbrace{\int \sin x \, e^x \, dx}_{\text{IBP again...}}$

$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$

$u = e^x$ $\quad\quad$ $v = -\cos x$

$du = e^x \, dx$ $\quad\quad$ $dv = \sin x \, dx$

thus...

$$\int e^x \cos x \, dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\Rightarrow 2 \int e^x \cos x \, dx = e^x (\sin x + \cos x) + C$$

$$\Rightarrow \int e^x \cos x \, dx = \frac{e^x}{2} (\sin x + \cos x) + \frac{C}{2}$$

$\square$

Definite IBP:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

EXAMPLE: $\displaystyle\int_0^4 x\cdot\frac{1}{e^x}\,dx = \frac{-x}{e^x}\Big|_0^4 - \int_0^4 \frac{-1}{e^x}\,dx = *$

$u = x \qquad V = \dfrac{-1}{e^x}$

$du = dx \qquad dV = \dfrac{1}{e^x}\,dx$

$\displaystyle * = \frac{-x}{e^x}\Big|_0^4 + \int_0^4 \frac{1}{e^x}\,dx = \frac{-x}{e^x}\Big|_0^4 + \frac{-1}{e^x}\Big|_0^4$

$\displaystyle = \left(\frac{-4}{e^4} - 0\right) + \left(\frac{-1}{e^4} + \frac{1}{1}\right) = \boxed{1 - 5/e^4}$ $\boxtimes$

We generally pick u so that du becomes simpler.

§ IBT TWICE

$x^2 \to 2x \to 2$ $\qquad e^x \to e^x \to e^x$ (will never change).

$$\int x^2 e^x \, dx = ⊛$$

$u = x^2 \qquad v = e^x$

$du = 2x\,dx \qquad dv = e^x\,dx$

$$⊛ = x^2e^x - 2\underbrace{\int e^x x \, dx}_{IBP} = x^2e^x - 2\left[xe^x - \int e^x dx\right]$$

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x\,dx$

$$= x^2e^x - 2xe^x + 2e^x + 2c.$$

EXERCISES

Note: You may not need IBP.

EX: $\int_0^1 x\sqrt{1-x}\,dx$

EX: $\int \frac{(\ln x)^3}{x}\,dx$

EX: $\int x^2 \frac{1}{e^x}\,dx$

EX: $\int_0^1 x(1-x)^{\frac{1}{2}}\,dx$

EX: $\int z(\ln z)^2\,dz$

EXAMPLE: $\int_1^2 \frac{\ln(x)}{x^2}\,dx = (*)$

$u = \ln x$ $\qquad$ $v = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$du = \frac{1}{x}\cdot dx$ $\qquad$ $dv = \frac{1}{x^2}\,dx$

$$(*) = \ln x \cdot \left(\frac{-1}{x}\right) - \int \left(\frac{-1}{x}\right)\cdot \frac{1}{x}\,dx$$

$$= \frac{-\ln x}{x} + \int \frac{1}{x^2}\,dx = \frac{-\ln x}{x} - \frac{1}{x} + C$$

$$= \frac{1}{x}(-\ln x - 1)$$