# RECAP

## FTC P1:

$$F(x) = \int_a^x f(t)\,dt \Rightarrow F'(x) = f(x)$$

that is...

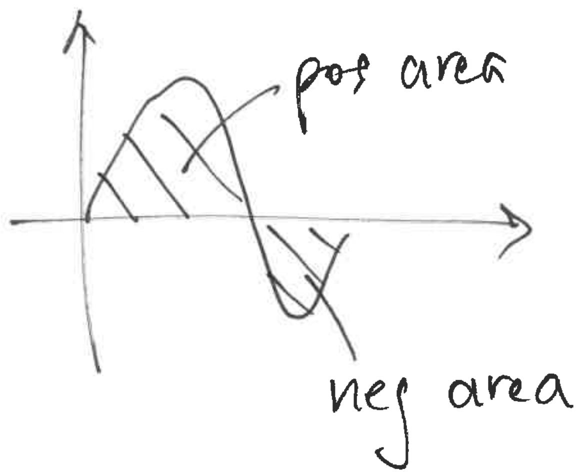
$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

## FTC P2:

$$F' = f \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

new? notation.

## NEGATIVE AREA:




## NOTATION:

$\int f\,dx$ is the "indefinite integral" equal/equivalent to the antiderivative of $f$.

## SUBSTITUTION: ~~[illegible]~~ (Reverse Chain Rule)

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

w/ $u = g(x)$.

**EXAMPLE:** $\int x\sqrt{2x+1}\, dx = *$

$u = 2x+1 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2} du = dx$

$\Downarrow$

$x = \frac{u-1}{2}$

$$* = \int \frac{u-1}{2}\sqrt{u} \cdot \frac{1}{2}\, du = \frac{1}{4}\int (u-1)\sqrt{u}\, du$$

$$= \frac{1}{4}\int u^{3/2} - u^{1/2}\, du = \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + c\right)$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C.$$

# § DEFINITE INTEGRAL SUB

<u>Thm</u> $$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

i.e. sub. must also be performed on the boundaries

<u>EXAMPLE:</u> $$\int_{x=-1}^{x=1} 3x^2(x^3+1)^{\frac{1}{2}}dx = \circledast$$

dx means samples along x.

<u>SUB METHOD 1:</u> Adjust boundaries

$$u = x^3+1 \Rightarrow du = 3x^2dx$$

$$x = -1..1 \Rightarrow u = (-1)^3+1 .. (1)^3+1 = 0..2$$

$$\circledast = \int_{u=0}^{u=2} u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}}\Big|_0^2 = \frac{2}{3}\left(2^{3/2} - 0^{3/2}\right)$$

$$= 2/3(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

SUB METHOD 2: Revert back to original variable and use old bounds

$$\int 3x^2(x^3+1)^{\frac{1}{2}}\,dx = \circledast$$

$$u = x^3+1 \Rightarrow du = 3x^2\,dx$$

$$\circledast = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(x^3+1)^{3/2} + C$$

$$\Rightarrow \int_{-1}^{1} 3x^2(x^3+1)^{\frac{1}{2}}\,dx = \frac{2}{3}(x^3+1)^{\frac{3}{2}}\Big|_{-1}^{1}$$

$$= \frac{2}{3}(2)^{\frac{3}{2}} - \frac{2}{3}(-1+1)^{\frac{3}{2}} = \frac{2}{3}\sqrt{8} = \frac{4\sqrt{2}}{3}$$

Recall: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

EXAMPLE: $\int_{-\pi/4}^{\pi/4} \tan\theta\,d\theta = \int_{-\pi/4}^{\pi/4} \frac{\sin\theta}{\cos\theta}\,d\theta = \circledast$

$$u = \cos\theta \Rightarrow u = \cos\frac{-\pi}{4} \ldots \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \ldots \frac{\sqrt{2}}{2}$$

$\circledast = -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{1}{u}\,du = 0$ because we are integrating over a single point.

Let us take the integral just to confirm

$$(*) = \int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{1}{u}\, du = -\ln|u| \Big|_{\sqrt{2}/2}^{\sqrt{2}/2} = \ln\left|\frac{\sqrt{2}}{2}\right| - \ln\left|\frac{\sqrt{2}}{2}\right| = 0.$$

Alternatively: Realize $\tan\theta$ is odd over $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and so we can exploit the symmetry.

Areas will cancel

More generally:

Thm Over an interval centered at zero $[-a, a]$

- $f$ even $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

- $f$ odd $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

EXAMPLE $\displaystyle\int_0^1 \frac{5r}{(4+r^2)^2}\,dr = \circledast$

$u = 4+r^2 \Rightarrow du = 2r\,dr \Rightarrow \frac{5}{2}du = r\,dr$

$r = 0..1 \rightarrow u = 4..5$

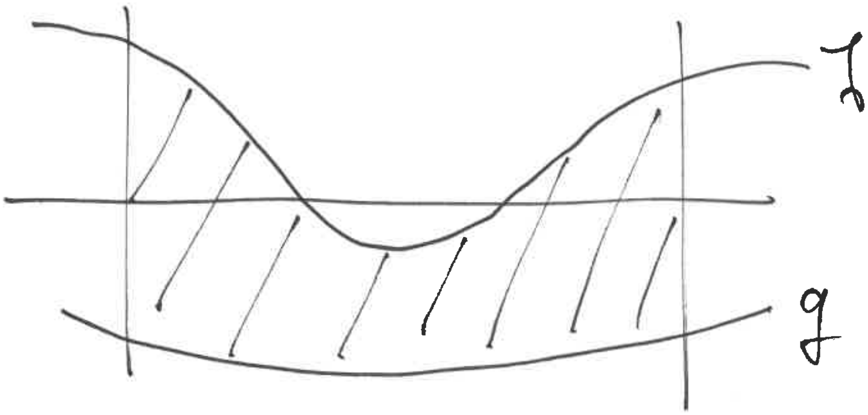
$$\circledast = \frac{5}{2}\int_4^5 \frac{1}{u^2}\,du = \frac{5}{2}\cdot \left.\frac{-1}{u^2}\right|_4^5$$

$$= \frac{5}{2}\left(\frac{-1}{5}+\frac{1}{4}\right) = \frac{5}{2}\left(\frac{5}{20}-\frac{4}{20}\right) = \frac{5}{40} = \frac{1}{8}$$

# § AREA BETWEEN CURVES

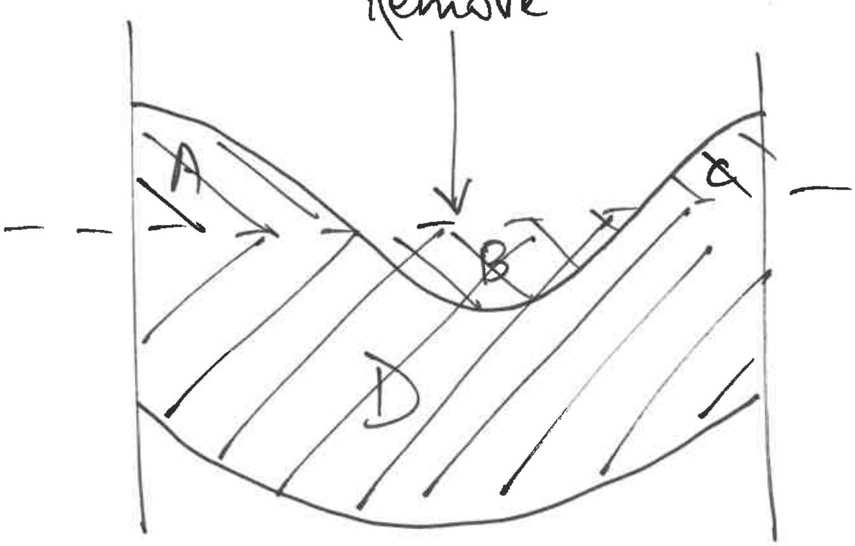


How can we obtain the following area?




i.e. "the area bounded by the curves $f$ & $g$."

~~don't double count B!~~
Remove




$A, B, C, D > 0$

Bounded Area = $A + C + D - B$ and,

$$\int_a^b f(x)\,dx = A - B + C,$$

$$\int_a^b g(x)\,dx = -D$$

<u>Notice</u>: $$\int_a^b f - g\,dx = \int_a^b f\,dx - \int_a^b g\,dx$$

$$= (A - B + C) - (-D) = A + C + D - B$$

$$= \text{Bounded Area}$$

<u>Def$^{\underline{n}}$</u> The "area bounded by curves $f$ and $g$" over $[a,b] \subseteq \mathbb{R}$ is given by:

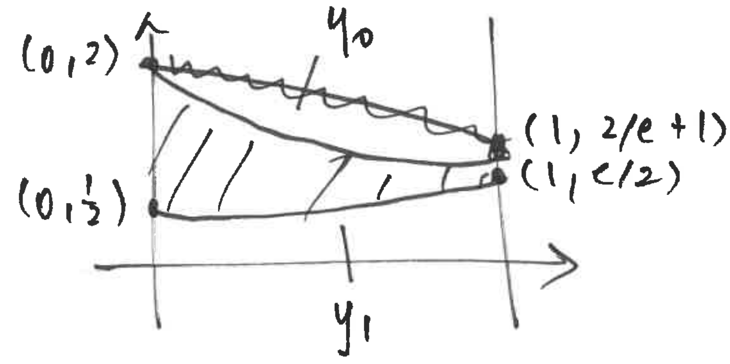
$$\int_a^b f(x) - g(x)\,dx$$

<u>provided</u> $f(x) \geq g(x)$ over $[a,b]$.

EXAMPLE: Find the area bounded by

$$y_0 = 2/e^x + x \quad \text{and} \quad y_1 = e^x/2$$

NOTE:




Supposing we dont have the graph we would need to argue $2/e^x + x \geq e^x/2$ for $x \in [0,1]$.

$$\text{Bounded Area} = \int_0^1 \left(\frac{2}{e^x} + x\right) - \left(\frac{e^x}{2}\right) dx$$

$$= \frac{-2}{e^x} + \frac{x^2}{2} - \frac{e^x}{2} \Big|_0^1$$

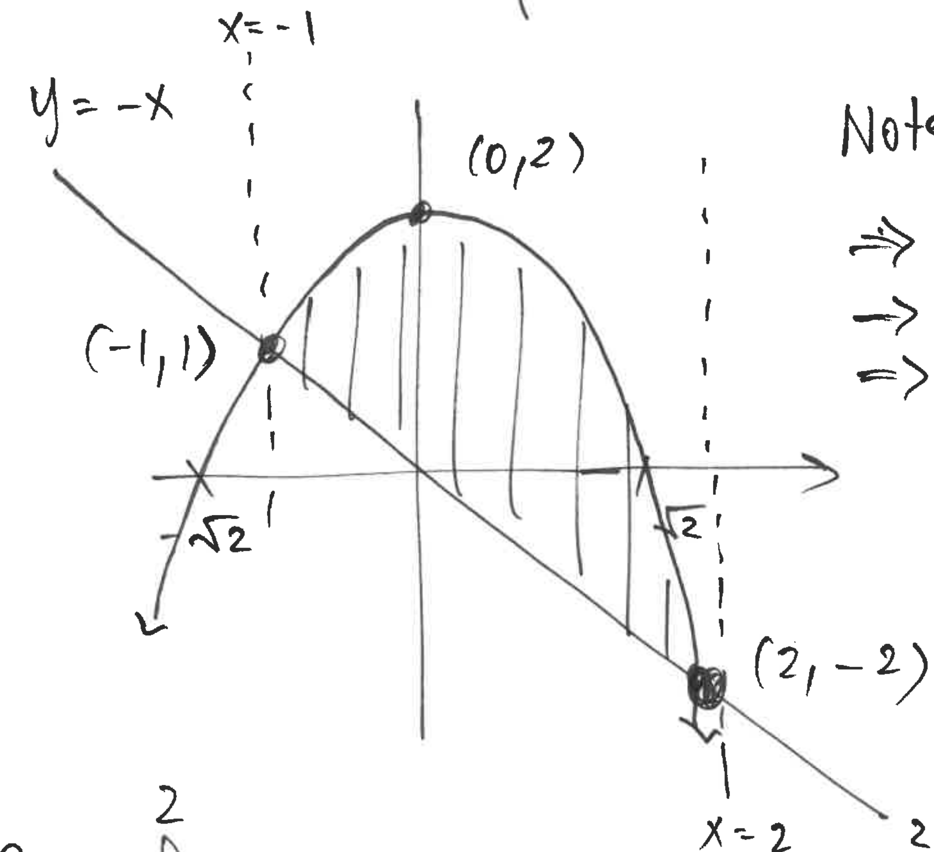
$$= \left(\frac{-2}{e^1} + \frac{1}{2} - \frac{e^1}{2}\right) - \left(\frac{-2}{1} + 0 - \frac{1}{2}\right)$$

$$= 3 - 2/e - e/2.$$

EXAMPLE: Find area enclosed by

$$y_0 = 2 - x^2 \quad \& \quad y_1 = -x$$

Here we are not given an interval because these curves form a ~~na~~ natural enclosure).




Note: $y_0 = y_1$

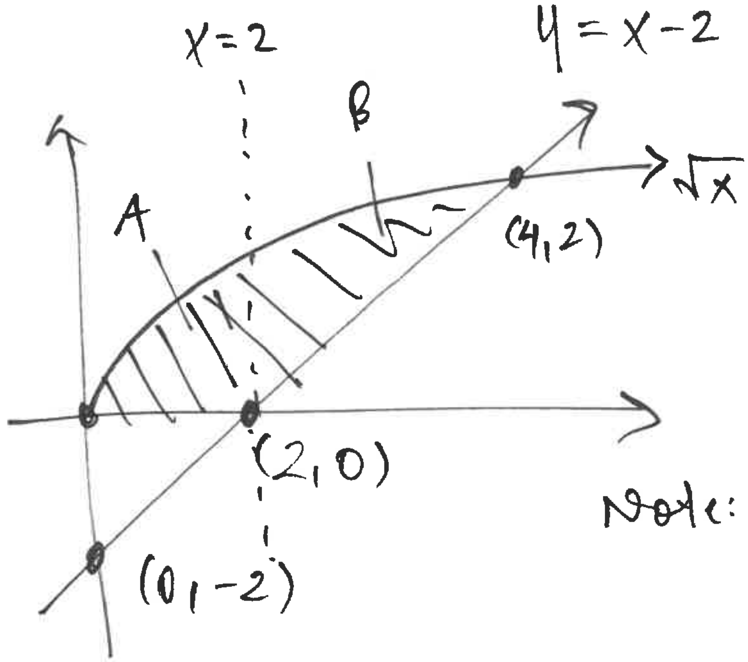
$$\Rightarrow 2 - x^2 = -x$$
$$\Rightarrow x^2 - x - 2 = 0$$
$$\Rightarrow (x-2)(x+1) = 0$$
$$\Rightarrow x = -1, 2.$$

$$BA = \int_{-1}^{2} (2-x^2) - (-x)\,dx = \int_{-1}^{2} 2 + x - x^2\,dx$$

$$= 2x + x^2/2 - x^3/3 \Big|_{-1}^{2} = \left(4 + 2 - 8/3\right) - \left(-2 + \tfrac{1}{2} + \tfrac{1}{3}\right)$$

$$= 8 - 9/3 - 1/2 = 8 - 3 - \tfrac{1}{2} = 9/2$$

EXAMPLE: Find BA in 1st quadrant bounded above by $y = \sqrt{x}$ and below by $y = x - 2$.




Note: $\sqrt{x} = x - 2$ when $x = 4$.

$$A = \int_0^2 \sqrt{x}\,dx = \frac{2}{3} x^{3/2} \Big|_0^2 = \tfrac{2}{3}\left(\sqrt{8} - 0\right) = \tfrac{4}{3}\sqrt{2}$$

$$B = \int_2^4 \sqrt{x} - x + 2\,dx = \frac{2}{3}x^{3/2} - x^2/2 + 2x \Big|_2^4$$

$$= \left(\frac{2}{3}4^{\frac{3}{2}} - \frac{4^2}{2} + 8\right) - \left(\frac{2}{3}2^{\frac{3}{2}} - 2 + 4\right)$$

$$= \left(\frac{2}{3}2^3 - 8 + 8\right) - \left(\frac{2}{3}\sqrt{8} + 2\right)$$

$$= \frac{2}{3}(8-\sqrt{8}) - 2 = \frac{16}{3} - \frac{4}{3}\sqrt{2} - 2$$

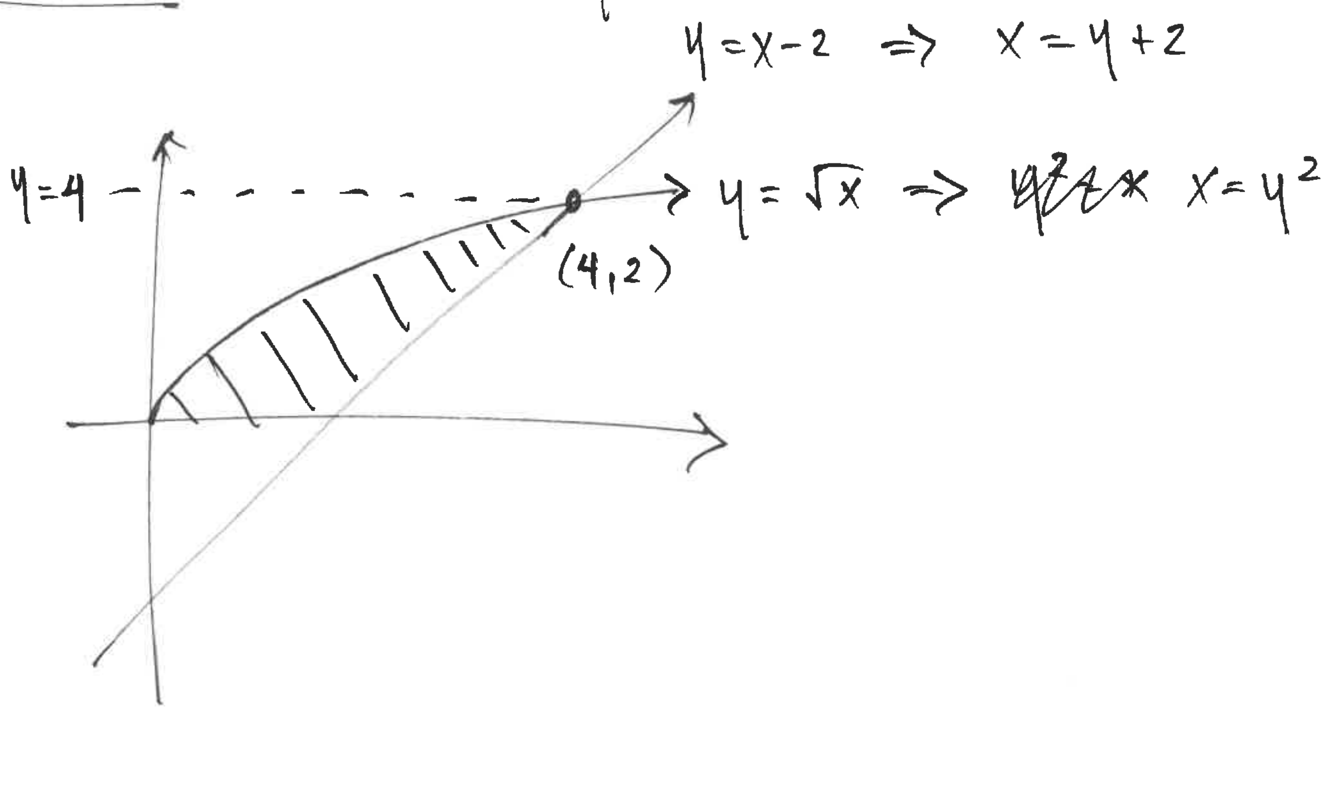
$$= \frac{10}{3} - \frac{4}{3}\sqrt{2}$$

$$\Rightarrow A + B = \text{Bounded Area} = \frac{4}{3}\sqrt{2} + \frac{10}{3} - \frac{4}{3}\sqrt{2}$$

$$= 10/3$$

This example would have been easier had we integrated along the y-axis.

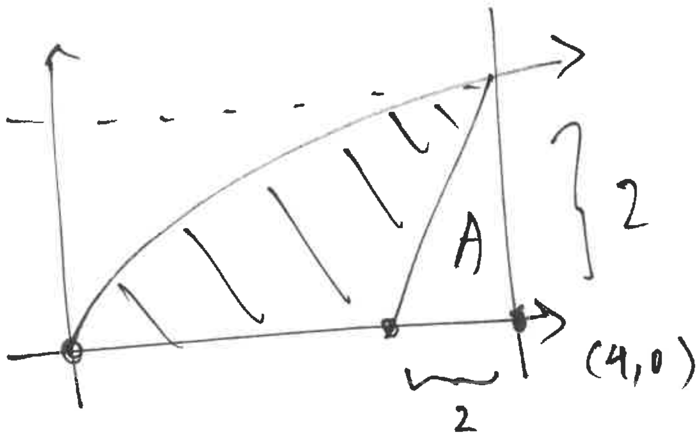
EXAMPLE: Same setup as last.



$$BA = \int_0^2 (y+2) - (y^2)\,dy = \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_0^2$$

$$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - (0+0-0) = 10/3$$

We could have also exploited geometry:




Note $A$ = area of triangle $= \frac{1}{2} 2 \cdot 2 = 2$

$$\text{Bounded Area} = \int_0^4 \sqrt{x}\,dx - 2$$

$$= \frac{2}{3} x^{3/2}\Big|_0^4 - 2$$

$$= \frac{2}{3} 2^3 - 2$$

$$= 10/3.$$

EXERCISE:

- $\displaystyle\int_0^3 \sqrt{y+1}\,dy$
- $\displaystyle\int_{\pi/6}^{\pi/3} (1-\cos^3 3t)\sin 3t\,dt$
- $\displaystyle\int_{-1}^{1} \frac{5r}{(4+r^2)^2}\,dr$
- $\displaystyle\int_0^{\pi/3} \tan^2\theta\,\cos\theta\,d\theta$

EXERCISE: Find enclosed regions area:

- $y = x^2 - 2, \quad y = 2$
- $y = 7 - 2x^2, \quad y = x^2 + 4$
- $x = 3\sin y\sqrt{\cos y}, \quad x = 0$ over $y \in [0, \frac{\pi}{2}]$
-