§ MEA - CULPA

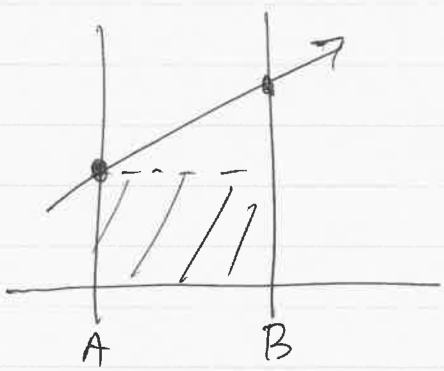
Mistake from yesterday. I wrote

"Curvature governs over/under estimations!"

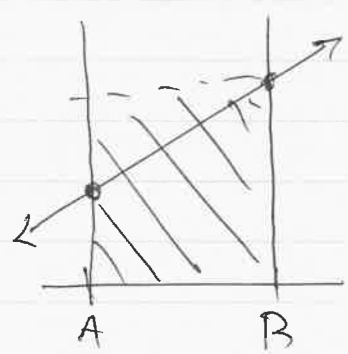
This is WRONG! ... sorry ~~

Increasing/Decreasing, i.e. the first derivative, governs the over/under estimations.

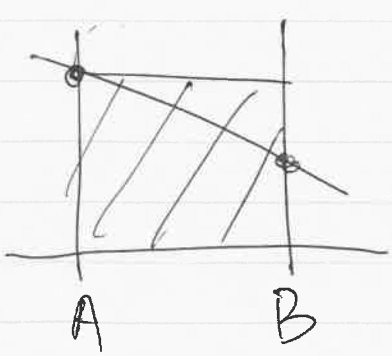
EXAMPLE



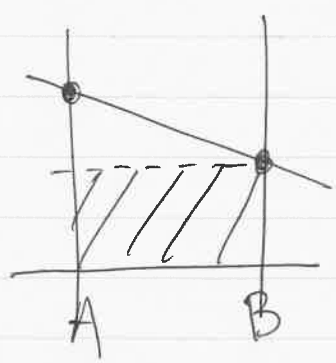
inc. LE under



inc. RE over



dec LE over



dec. RE under

§ Limits of Finite Sums

Recall from yesterday:

$$A = \{3, 7, -1\}$$

$$\sum_{a \in A} a = 3 + 7 + (-1)$$

We can also number the elements of A:

$$A = \{a_0, a_1, \cdots, a_N\}$$

then $$\sum_{K=0}^{N} a_K = a_0 + a_1 + \cdots + a_N$$

Also: $$\sum_{K=-3}^{2} K = (-3) + (-2) + (-1) + 0 + 1 + 2$$

supposing $f: \mathbb{R} \to \mathbb{R}$

$$\sum_{K=-3}^{2} f(K) = f(-3) + f(-2) + f(-1) + f(0) + f(1) + f(2)$$

EXERCISE

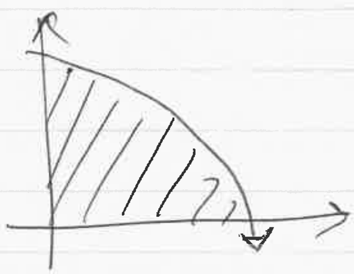
Supposing $c \in \mathbb{R}$ is const.

- $$\sum_{K=0}^{2} 7 = \overset{0}{7} + \overset{1}{7} + \overset{2}{7} = 21$$
- $$\sum_{a \in A} c \cdot a = c \sum_{a \in A} a$$

$\mathcal{A}$: "script/fancy" A.
(too distinguish from "A" in $\underset{A}{+}$ — $\underset{B}{+}$)

Todays goal:

Find the precise area under the curve of $f(x) = 1 - x^2$ over $[0,1]$.



f is decreasing over $[0,1]$ thus:

LE are over approx

RE are under approx.

$\Rightarrow$ RE approx $\leq$ $\underbrace{\text{Precise Area}}_{=\mathcal{A}}$ $\leq$ LE approx

As warmup ... w/ 3 squares

$\Delta x \quad \Delta x \quad \Delta x$

$0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1$

$\Delta x = \frac{1}{3}$

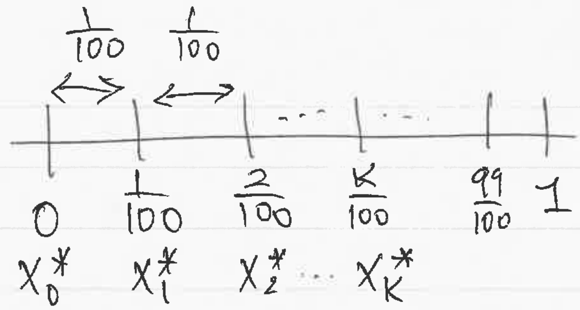
$LE = \{0, \frac{1}{3}, \frac{2}{3}\}$ $\qquad$ $RE = \{\frac{1}{3}, \frac{2}{3}, 1\}$

$$\Rightarrow \sum_{x^* \in RE} \Delta x f(x^*) \leq \mathcal{A} \leq \sum_{x^* \in LE} \Delta x f(x^*)$$

$$= \sum_{x^* \in RE} \frac{1}{3}(1 - (x^*)^2)$$

$$\Rightarrow \quad 0.482 \leq \mathcal{A} \leq 0.815$$

w/ 100 squares



$\Delta x = \frac{1}{100}$ (the distance inbetween two samples if we space them equally.)

$LE = \{ 0/0, 1/100, \ldots, 99/100 \}$

$RE = \{ 1/100, \ldots, 100/100 \}$

$$\Rightarrow \sum_{x^* \in RE} \frac{1}{100}\left(1 - (x^*)^2\right) \leq A \leq \sum_{x^* \in LE} \frac{1}{100}\left(1 - (x^*)^2\right)$$

... this is where computers come in handy...
In my CSC 108 class you would learn how to do:

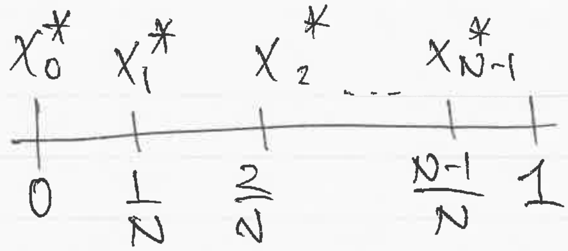
```
>>> RE = [k/100 for k in range(100)]
>>> sum( (1-x**2)/100 for x in RE )
0.662
```

by the same method LE gives 0.671

$\Rightarrow \quad 0.662 \leq A \leq 0.671$

5.

w/ N-squares



$x_K^* = \frac{K}{N}$

$\Delta x = \frac{1}{N}$

$LE = \{0, \dots, \frac{N-1}{N}\} = \{x_0^*, \dots, x_{N-1}^*\}$

$$\Rightarrow \sum_{x^* \in LE} \frac{1}{N}(1-(x^*)^2) = \sum_{K=0}^{N-1} \frac{1}{N}\left(1-\left(\frac{K}{N}\right)^2\right)$$

$$= \sum_{K=0}^{N-1}\left[\frac{1}{N} - \frac{K^2}{N^3}\right] = \sum_{K=0}^{N-1}\frac{1}{N} - \sum_{K=0}^{N-1}\frac{K^2}{N^3}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} 1 - \frac{1}{N^3}\underbrace{\sum_{K=1}^{N-1} K^2}_{} = \cdots \quad (*)$$

note the index change only remove 0 sum

<u>NOTE</u>: $\sum_{K=1}^{N} K^2 = \frac{N(N+1)(2N+1)}{6}$

$$(*) = \frac{1}{N}\cdot N - \frac{1}{N^3}\cdot\frac{(N-1)(N)(2N-1)}{6} = ~~\text{Area w/ N rectangles}~~$$

Area w/ N-rectangles

$$= 1 - \frac{2}{6} - \frac{3}{6N} + \frac{1}{6N^2} = ~~\text{STUCK cannot get rid of N}~~$$

But <u>calculus</u>! Let's use $N \to \infty$ rectangles. ~~STUCK~~

LE area is exactly

$$\lim_{N \to \infty} 1 - \frac{2}{6} - \frac{3}{6N} + \frac{1}{6N^2} = 1 - \frac{1}{3} = \frac{2}{3}$$

By the same argument we find RE area is $2/3$ as well.

Thus $\quad 2/3 \leq A \leq 2/3 \implies A = 2/3 \approx 0.666$

exactly.

This geometric approach was cumbersome — we want an algebraic way as well. ~~(Later)~~.
(Later: FTC).

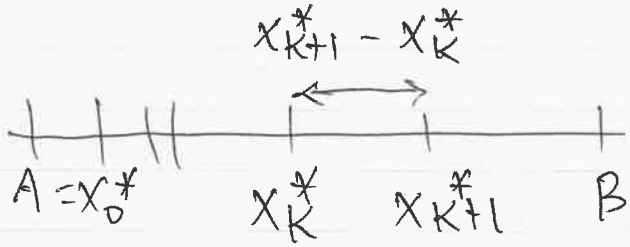
§ Riemann Sum

The sums we have been working w/ are called Riemann Sums.

Let's generalize...

CASE Random samples

Δx is not constant



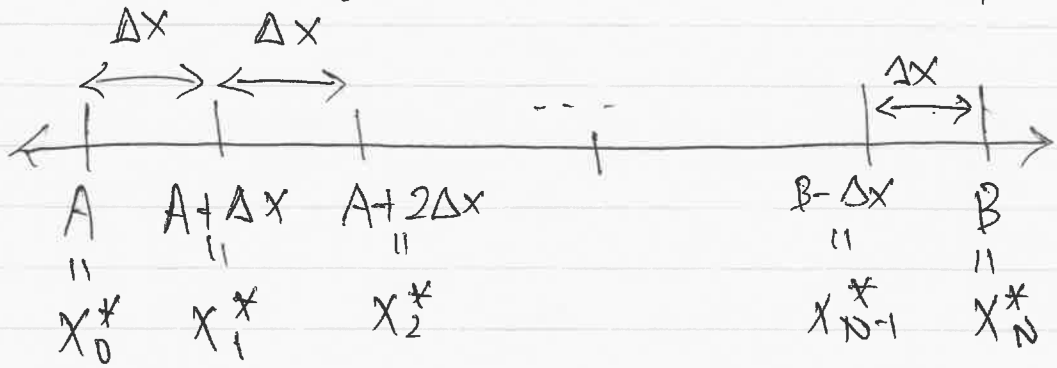
$LE = \{ A = x^*_0, x^*_1, \dots, x^*_{N-1} \}$

$RE = \{ x^*_1, \dots, x^*_N \}$ ← there is no pattern for x^*_k, it's random

LE approx: $\displaystyle\sum_{k=0}^{N-1} (x^*_{k+1} - x^*_k) f(x^*_k)$

RE approx: $\displaystyle\sum_{k=1}^{N} (x^*_k - x^*_{k-1}) f(x^*_k)$

<u>CASE</u> N-equally spaced samples $\Delta x = \frac{B-A}{N}$



$$A = x_0^*,\quad A+\Delta x = x_1^*,\quad A+2\Delta x = x_2^*,\quad \ldots,\quad B-\Delta x = x_{N-1}^*,\quad B = x_N^*$$

$$LE = \{ A = x_0^*, \cdots, x_{N-1}^* \}$$

$$RE = \{ x_1^*, \cdots, x_N^* = B \}$$

LE approx:
$$\sum_{K=0}^{N-1} \Delta x \, f(x_K^*) = \sum_{K=0}^{N-1} \frac{b-a}{N} f(a + K\Delta x)$$

RE approx:
$$\sum_{K=1}^{N} \Delta x \, f(x_K^*) = \sum_{K=1}^{N} \frac{b-a}{N} f(a + K\Delta x)$$

<u>EXERCISE</u> i Find the <u>exact</u> area under the line/curve ~~over [0,1]~~ $y = 2x$ over $[0,1]$. by reading it off the graph (its a triangle)

ii Find the area using Riemann sums and confirm you get i's answer.

Summary:

The area under the curve $f(x)$ over $[a, b]$ is given by

$$\lim_{N \to \infty} \sum_{K=0}^{N-1} f(X_K^*) \cdot \Delta X = \int_a^b f(x)\,dx$$