§ Inverse Functions and Logarithms

MOTIVATION: Consider $y = 2x$. For what x is $y = 7$?

A: $x = \frac{7}{2}$.

Letting $y = f(x)$ indicates we want to "give" x and "get" y.

Conversely, letting $x = g(y)$ indicates we want to "give" y to "get" x.

Here: $y = 2x$ OR $f(x) = 2x$ OR $y = 2 \cdot g(y)$

Notice: $y = 2 \cdot g(y) \Rightarrow g(y) = y/2$,

$g(7) = \frac{7}{2}$, and

$f(g(7)) = f(\frac{7}{2}) = 7$.

Generally f and g are said to be "inverses" of one another.

Def$^{\underline{n}}$ Inverse Mapping

Let $f: A \to B$ be a mapping. The <u>inverse mapping</u>:

$$f^{-1}: B \to A$$

has the property:

$$f(a) = b \iff f^{-1}(b) = a.$$

<u>EXAMPLE</u> Let $f(x) = 3x + 7$.

Find $f^{-1}(28)$: By guessing $f(7) = 3 \cdot 7 + 7 = 28$.
so $f^{-1}(28) = 7$.

Find $f^{-1}(x)$. Let $x = g(y)$

$$y = 3 \cdot g(y) + 7 \implies g(y) = \frac{y-7}{3}$$

So $f^{-1}(x) = g(x) = \dfrac{x-7}{3}$.

<u>NOTICE</u> • $G(f) = \{(x,y) : y = f(x) \text{ AND } x \in \operatorname{dom} f\}$

$\implies G(f^{-1}) = \{(y,x) : y = f(x) \text{ AND } x \in \operatorname{dom} f\}$

i.e. the graph of f^{-1} is the reflection about $y=x$ of f.

NOTICE

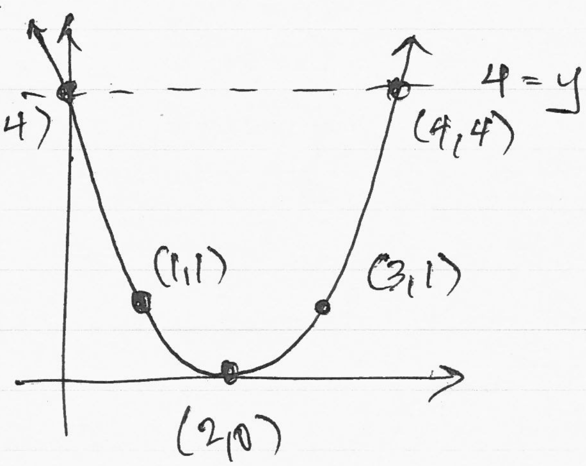
- $f^{-1}(b)=a \Rightarrow f(f^{-1}(b)) = f(a) \rightarrow f(a) = f(b)$

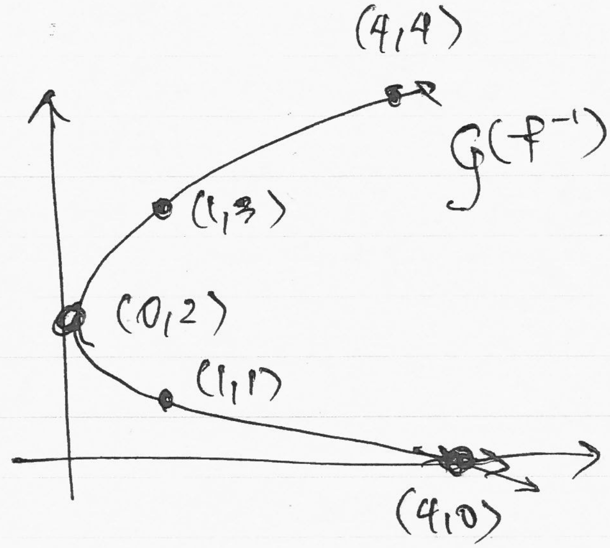
EXAMPLE $f(x) = (x-2)^2$

$(2,0),\ (3,1),\ (1,1),\ (4,4),\ (0,4) \in G(f)$

$\rightarrow (0,2), (1,3), (1,1), (4,4), (4,0), \in G(f^{-1})$

$G(f)$:





$\text{dom } f = \mathbb{R}$

$\text{rng } f = [0, \infty)$

$\text{dom } f^{-1} = [0, \infty)$

$\text{rng } f^{-1} = \mathbb{R}$

NOTE: $f : \text{dom } f \rightarrow \text{rng } f$

$f^{-1} : \text{rng } f \rightarrow \text{dom } f$

Clearly f^{-1} does not necessarily pass the vertical line test - even if f does.

That is to say: A function f does not always have an inverse function.

<u>QUESTION</u>: What is the pre-condition on f that ensures f^{-1} is a function as well?

<u>ANSWER</u>: f is "one-to-one" $\Rightarrow$ f^{-1} ~~is a~~ is a function.

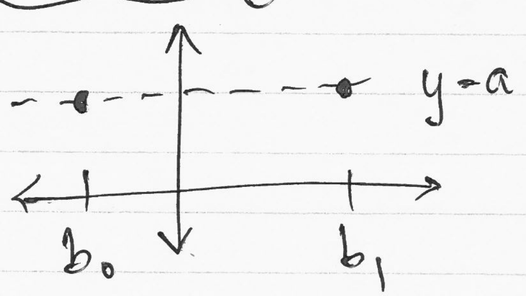
We can work out what this means "algebraically"

f^{-1} is a function when it satisfies:

$$(f^{-1}(a) = b_0 \text{ and } f^{-1}(a) = b_1) \Rightarrow b_0 = b_1$$

thus $f(b_0) = a$ and $f(b_1) = a \rightarrow b_0 = b_1$

<u>Geometrically</u>: This is impossible!



<u>English</u>: ~~Vertical~~ Horizontal line test!

Def$^{\underline{n}}$ A function that satisfies the vertical line test is said to be "<u>one-to-one</u>".

§ <u>LOGARITHMS</u>

<u>QUESTION</u> Let $f(x) = 10^x$. What is $f^{-1}(100)$?

$$f^{-1}(100) = a \iff f(a) = 10^a = 100 \iff a = 2.$$

The exponential functions have special inverses:

Here $\log_{10} 100 = 2 \iff 10^2 = 100.$

Def$^{\underline{n}}$ Logarithm

$$\log_b a = x \iff b^x = a.$$

<u>EXAMPLE</u> $\log_{10} 1000 = 3 \iff 10^3 = 100$

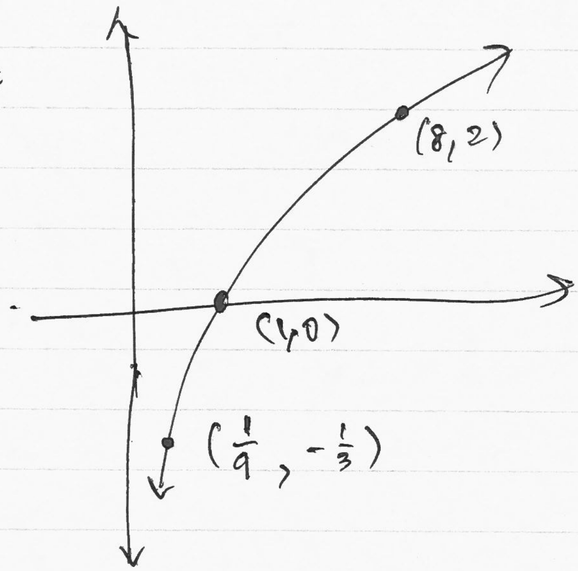
$$\log_2 8 = 3 \iff 2^3 = 8$$

$$\log_2 2 = \sqrt{2} \iff \sqrt{2}^2 = 2$$

$$\log_3 \frac{1}{9} = -2 \iff 3^{-2} = \frac{1}{9}$$

EXAMPLE:

$y = \log_2 x$



LAW OF LOGS

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b x^r = r\log_b x$$

EXAMPLE

$\log_2 80 - \log_2 5.$

$$= \log_2 2^3 \cdot 2 \cdot 5 - \log_2 5$$

$$= \log_2 \frac{2^4 \cdot 5}{5}$$

$$= \log_2 2^4$$

$$= 4\log_2 2 = 4.$$

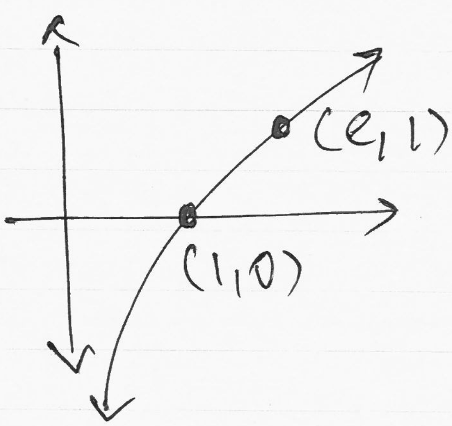
Def$^{\underline{n}}$ Natural Logarithm

$$\log_e x = \ln x \qquad x \in \mathbb{R}.$$

NOTE

- $\ln x = y \iff e^y = x$
- $\ln(e^x) = x \qquad x \in \mathbb{R}$
- $e^{\ln x} = x \; : \; x > 0$
- $\ln e = 1$

EXAMPLE



EXAMPLE Solve:

$$e^{5-3x} = 10 \Rightarrow \ln e^{5-3x} = \ln 10$$
$$\Rightarrow (5-3x)\ln e = \ln 10$$
$$\Rightarrow 5 - 3x = \ln 10$$
$$\Rightarrow x = \frac{5 - \ln 10}{3}$$

We normally use ln when reducing /solving exp-eqns

<u>Prop$^{\underline{n}}$</u> $\log_a x = \dfrac{\ln x}{\ln a}$

<u>Proof</u>: $y = \log_a x \iff a^y = x \iff y \ln a = \ln x$

$\iff y = \dfrac{\ln x}{\ln a} \iff \log_a x = \dfrac{\ln x}{\ln a}$.

EXERCISE: Find $a > b > 0$ such that

$$a^b = b^a.$$

EXERCISE: Find a formula of the inverse func $f^{-1}(x)$ for

① $f(x) = \dfrac{4x-1}{2x+3}$ ③ $y = \ln(x+3)$.

② $f(x) = e^{x^3}$

CHECK Your answer by confirming

$$f(f^{-1}(x)) = x.$$

EXERCISE: Solve for x

① $2^{x-5} = 3$ ② $\ln x + \ln(x-1) = 1$

EXERCISE: $g(x) = 3 + x + e^x \Rightarrow g^{-1}(4) = ?$