# Practice Problems For Mathematics 4B03

**Question 1** : (Carmo page 61)

An *n-dimensional differentiable manifold with* (regular) *boundary* is a set $M$ and a family of injective maps $f_\alpha : U_\alpha \subset H^n \to M$ of open sets of $H^n$ into $M$ such that:

1. $\bigcup_\alpha f_\alpha(U_\alpha) = M$
2. For all pairs $\alpha$, $\beta$ with $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \phi$ the sets $f_\alpha^{-1}(W)$ and $f_\beta^{-1}(W)$ are open sets in $H^n$ and the maps $f_\beta^{-1} \circ f_\alpha$, $f_\alpha^{-1} \circ f_\beta$ are differentiable.
3. The family $\{(U_\alpha, f_\alpha)\}$ is maximal relative to (1) and (2).

(i) $S^4$ is an example of a four-dimensional compact manifold with boundary.

(ii) The mobius band is an example of a non-orientable two-dimensional compact manifold with boundary.

(iii) A cone with a point removed or $H^n$, the half plane with dimension $n$, are both examples of an orientable non-compact manifold with boundary.

**Question 2** : (Hitchin page 23)

A *vector field on a manifold* $M$ is a smooth map

$$X : M \to TM = \bigcup_{a \in M} T_a$$

such that

$$p \circ X = id_M.$$

- $T_a$ when $a \in M$ is the dual space of the quotient space $T_a^* = C^\infty(M)/Z_a$
- $C^\infty(M)$ is all the $C^\infty$ function on $M$. Recall that a function is $C^\infty$ if it has derivatives of all orders.
- $p : TM \to M$ is the projection map which assigns $X_a \in T_a M$ to $a$. $p$ has the property that it is smooth with surjective derivative.

(i) For $a, b \in \mathbb{H}$ (the unit quaternions) with the regular definitions of $i, j$ and $k$ the following 6 vector fields are everywhere linearly independent on $S^3 \times S^3$.

$$\begin{array}{ll} < ia, ib > & < ia, jb > \\ < ja, jb > & < ia, kb > \\ < ka, kb > & < ja, kb > \end{array}$$

(ii) Pictured on Camo page 101 for $I = -2$ where $I$ is the index.

(iii) By Gauss-Bonet theorem there are no vector fields on $S^2$ which have no zeros. The explanation is as follows: Gauss Bonet Theorem states that

$$2\pi\chi(M) = 2\pi \sum_{p_i \in A} I(p_i) = \iint_M k dA$$

where $\chi$ is the Euler characteristic, $A = \{p : V(p) = 0\}$ where $V$ is any vector field, and $I$ is the index of $p_i$.

We know that $\chi(S^2) = 2$ however in order to have a vector filed $V$ with no zeros we must have that $2\pi\chi(S^2) = 0$ which is impossible.

**Question 3** : (Garrity page 122)

Let $M$ be a manifold. An exterior differential $k$-form on $M$ is a map $\omega$ such that

$$\omega : M \longrightarrow \wedge^k (T_p M)*,$$

Written:

$$\omega(p) = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k}(p)(dx^{i_1} \wedge \ldots \wedge dx^{i_k})_p$$

for $i_j \in \{1 \ldots n\}$, where each $a_{i_1 \ldots i_k}$ is differentiable.

(i) There are no closed 2-forms on $S^3$ that are not exact.

(ii) A simple closed 1-form on the torus $T^2$, using the canonical parametrization in euclidian space, is:

$$\omega = d\theta$$

(iii)

**Question 4** : (mathworld)

A *lie group* is a differentiable manifold obeying the group properties and that satisfies the additional condition that the group operations are differentiable.

The simplest examples of Lie groups are one-dimensional. Under addition, the real line is a Lie group. After picking a specific point to be the identity element, the circle is also a Lie group. Another point on the circle at angle $\theta$ from the identity then acts by rotating the circle by the angle $\theta$ In general, a Lie group may have a more complicated group structure, such as the orthogonal group $O(n)$ (i.e., the orthogonal matrices), or the general linear group (i.e., the invertible matrices).

A *Lie Group* is a differentiable manifold $G$ that is also a group in the algebraic sense, with multiplication $m : G \times G \longrightarrow G$ and inversion $i : G \longrightarrow G$, both differentiable, given by:

$$m(g, h) = gh \qquad\qquad i(g) = g^{-1}$$

Look at $SO(3) = \{A \in O(3) : det(A) = 1\}$:

This is clearly a group under matrix multiplication since $I_3 \in SO(3)$, matrix multiplication is associative, and for $A, B \in SO(3)$:

$$(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1} \qquad det(AB) = det(A)det(B) = 1$$

and

$$A^{-1} = A^T \in SO(3) \qquad \text{since} \qquad det(A) = det(A^T) \qquad \text{and} \qquad A^T \in O(3).$$

In $SO(3)$, multiplication is differentiable because the matrix entries of $AB$ are linear polynomials of the entries of $A$ and $B$, and inversion is differentiable because Cramer's rule expresses entries of $A^{-1}$ as rational functions of entries of $A$.

Look at $T^3 = S^1 \times S^1 \times S^1$:

We can easily see that $S^1$ is a lie group by simply imbedding it in the complex plane. Then $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. This is a group under complex multiplication, and $m(e^{i\theta_1}, e^{i\theta_2}) = e^{i(\theta_1+\theta_2)}$ and $i(e^{i\theta_1}) = e^{-i\theta_1}$ are both clearly smooth maps.

Now, if $G_1$ and $G_2$ are lie groups, $G_1 \times G_2$ is also a lie group with componentwise multiplication, so $T^3$ is a lie group.

Next, a basis for the tangent space of SO(3) at the identity is:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Since $SO(3)$ is a lie group, we can use the group action through left multiplication to cycle through the group, so our linearly independent vector fields are:

$$\varphi(a) = aE_1 \qquad \varphi(a) = aE_2 \qquad \varphi(a) = aE_3.$$

NOT DONE

**Question 5:** (Hitchin page 53)

An $n$-dimensional manifold is said to be *orientable* if it has an everywhere non-vanishing $n$-form $\omega$.

$\mathbb{R}p^n$ is orientable if $n$ is odd and nonorientable if $n$ is even. There is a map $\phi : S^n \to \mathbb{R}p^n$ defined as $\phi : x \mapsto [x]$ where $[x]$ is the equivalency class of $x \sim -x$ where the $\det \phi = (-1)^{n+1}$ so $S^n$ has an orientation which will carry over to $\mathbb{R}p^n$ if the $\det \phi = 1$. This implies that if $n$ is odd $\mathbb{R}p^n$ has an orientation.

Yes, all Lie groups have an orientation. Pick any orientation at the identity point and then move this orientation to any point $p$ by using the group operation.

**Question 6:** (mathworld)

Let $f : M \to N$ be a map between two compact, connected, oriented $n$-dimensional manifolds without boundary. Then $f$ induces a homomorphism $f_*$ from the homology groups $H_n(M)$ to $H_m(N)$, both canonically isomorphic to the integers, and so $f_*$ can be thought of as a homomorphism of the integers. The integer $d(f)$ to which the number 1 gets sent is called the degree of the map $f$.

There is an easy way to compute $d(f)$ if the manifolds involved are smooth. Let $x \in \mathbb{N}$, and approximate $f$ by a smooth map homotopic to $f$ such that $x$ is a "regular value" of $f$ (which exist and are everywhere by sard's theorem). By the implicit function theorem, each point in $f^{-1}(x)$ has a neighborhood such that restricted to it is a diffeomorphism. If the diffeomorphism is orientation preserving, assign it the number $-1$, and if it is orientation reversing, assign it the number $-1$. Add up all the numbers for all the points in $f^{-1}(x)$, and that is the $d(f)$, the degree of $f$. One reason why the degree of a map is important is because it is a homotopy invariant. A sharper result states that two self-maps of the $n$-sphere are homotopic iff they have the same degree. This is equivalent to the result that the $n$-th homotopy group of the $n$-sphere is the set $\mathbb{Z}$ of integers. The isomorphism is given by taking the degree of any representation.

One important application of the degree concept is that homotopy classes of maps from $n$-spheres to $n$-spheres are classified by their degree (there is exactly one homotopy class of maps for every integer $n$, and $n$ is the degree of those maps).

A map $F$ of degree one from the torus $T^2 = S^1 \times S^1$ to the sphere $S^2$ is given by:

$$F : (\theta, \phi) \mapsto (\theta', \phi')$$

where

$$T^2 = S^1 \times S^1 = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$$

and

$$S^2 = (\cos\theta'\cos\phi', \sin\theta'\cos\phi', \sin\phi')$$

Where $F^{-1}(\pi, 0) = (\pi, 0)$ and $sign(DF_{\pi,0}) = 1$ which implies that $degF = 1$.

**Question 7:** Since there are no exact 0-forms on $T^2$ or $S^2$, and the closed forms on $S^2$ and $T^2$ are functions with $df = 0$ (constants). We have that

$$H^0(T^2) \approx H^0(S^2) \cong \mathbb{R}$$

Since every circle on $S^2$ is contractible to a point and every 1-form on $S^2$ can be seen as a 1-form on a circle in $S^2$ because if you have a closed form $\omega$ on $S^2$ by Poincare's Lemma you may take $\omega = df_+$ for some function $f_+$ on the upper half of $\omega$ and $\omega = df_-$ for some function $f_-$ on the lower half. Since

$$\int_{\text{the equator}} \omega = 0$$

we have that

$$\oint df_+ = \oint df_- = 0 \Rightarrow \oint df_+ - df_- = 0$$

Which gives $d(f_+ - f_-) = 0$ so $f_+ = f_- + C$ for some constant $C$. So for $S^2$ $\omega = d(f_+)$ for some function $f$, so $\omega$ is exact.

**lemma:** every closed 1-form is exact. So there are no closed non-exact 1-froms $\Rightarrow H^1(S^2) = \{0\}$

$T^2$ on the other hand has 2-linenarly independent 1-froms $d\theta_1$, $d\theta_2$ where $T^2$ is paramaterized by $(e^{i\theta_1}, e^{i\theta_2}$.

Any closed one-form on $T^2$ differes from $c_1 d\theta_1 + c_2 d\theta_2 + ...$ $c_1, c_2 \in \mathbb{R}$ by an exact form. So $\{\text{closed forms/exact forms}\} \cong \mathbb{R}^2 \Rightarrow H^1(T^2) \approx \mathbb{R}^2$ and isomorphic to $H^0(T^2)$ and $H^0(S^2)$, so

$$H^2(T^2) \approx \mathbb{R} \approx H^2(S^2)$$

Now to show that every differentiable map $f : S^2 \to T^2$ has degree zero we do

$$deg(f) = \frac{\int_S^2 f^* \omega}{\int_T^2 \omega}$$

so $\omega$ is a closed form which is not exact because $deg(f) = \frac{0}{\text{something}} = 0$.

**Question 8:** .

To prove that $\mathcal{L}_x \phi = \iota_x d\phi + d\iota_x \phi$ where $\iota_x : \Omega^p \to \Omega^{p-1}$, $d : \Omega^p \to \Omega^{p+1}$, and $\mathcal{L}_x : \Omega^p \to \Omega^p$ we only need to check the formula on functions and 1-forms of the form $dx^i$. So we let

$$\alpha = \sum a_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}.$$

So since $\alpha$ is a p-form $\iota_x \alpha(v_1, ..., v_{p-1}) = \alpha(x, v_1, ..., v_{p-1})$ and

$$\mathcal{L}_x f = \lim_{t \to 0} \frac{\phi_t^*(f) - f}{t} = \frac{f \circ \phi_t - f}{t}.$$

So the directional derivative of $F$ in the direction of $x$ is given by $(\iota_x d + d\iota_x) f = \iota df = df(x) = x(f) = df(x)$. So $\mathcal{L}_x = \iota_x d + d\iota_x$which implies that $\mathcal{L}_x \phi = (\iota_x d + d\iota_x)\phi = \iota_x d\phi + d\iota_x \phi$.

To computer $\mathcal{L}_x \phi$ for $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ and $\phi = e^{-x^2-y^2-z^2} dx \wedge dy \wedge dz$ on $\mathbb{R}^2$ we realize that

$$= \mathcal{L}_x(f(x,y,z) dx \wedge dy \wedge dz)$$
$$= (\mathcal{L}_x f) dx \wedge dy \wedge dz + f d(\mathcal{L}_x x) \wedge dy \wedge dz + f dx \wedge d(\mathcal{L}_x y) \wedge dz + f dx \wedge dy \wedge d(\mathcal{L}_x z)$$

where $\mathcal{L}_x x = (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})x = x$ so

$$= \mathcal{L}_x f dx \wedge dy \wedge dz + 3f dx \wedge dy \wedge dz$$

we also note that

$$
\begin{aligned}
&= \mathcal{L}_x f \\
&= (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})e^{-x^2-y^2-z^2} + x(e^{-x^2-y^2-z^2})(-2x) + y(e^{-x^2-y^2-z^2})(-2y) + z(e^{-x^2-y^2-z^2})(-2z) \\
&= -(2x^2+2y^2+2z^2)e^{-x^2-y^2-z^2}
\end{aligned}
$$

so we complete the computation by doing

$$
\begin{aligned}
&= \mathcal{L}_x f dx \wedge dy \wedge dz + 3f dx \wedge dy \wedge dz \\
&= -(2x^2+2y^2+2z^2)e^{-x^2-y^2-z^2} dx \wedge dy \wedge dz + 3e^{-x^2-y^2-z^2} dx \wedge dy \wedge dz
\end{aligned}
$$

**Question 9:**

**Question 10:** (Garrity page 138)

*Stoke's Theory:* Let $M$ be an oriented $k$-dimensional manifold in $\mathbb{R}^n$ with boundary $\partial M$, a smooth $(k-1)$-dimensional manifold with orientation induced from the orientation of $M$. Let $\omega$ be a differential $(k-1)$-form. Then:

$$\int_M d\omega = \int_{\partial M} \omega.$$

This is the quantitative version of the intuition that the

average of a function on boundary = average of a derivative on interior.

We will define $S^-$ to be the sphere $x^2+y^2+z^2 \leq 1$ and $S^+$ to be the sphere $x^2+y^2+z^2 \leq 4$. and not that $\int_{S^2} P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy = \int (P,Q,R) \cdot \vec{n}$ where $\vec{n}$ is the normal to the sphere. Finally as given

$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{x^2+y^2+z^2}$$

**for** $S^+$

$$\int_{x^2+y^2+z^2=1} \frac{(x,y,z)}{4} \cdot \frac{(x,y,z)}{z} = \int_{S^2(2)} \frac{1}{2} = \frac{1}{2} vol(S^2(2)) = \frac{1}{2} 4\pi \cdot 4 = 8\pi$$

**for** $S^-$

$$\int_{x^2+y^2+z^2=1} (x,y,z) \cdot (x,y,z) = \int_{S^2} 1 = 4\pi$$

So

$$\int_{\partial M} = \int_{S^+} \omega - \int_{S^-} \omega = 8\pi - 4\pi = 4\pi$$

Now it is necessary to check $\int d\omega$ against Stokes Theorem.

$$\begin{aligned} d\omega &= 3\frac{dx \wedge dy \wedge dz}{x^2+y^2+z^2} - \frac{(2x^2+2y^2+2z^2)dx \wedge dy \wedge dz}{(x^2+y^2+z^2)^2} \\ &= 3\frac{dx \wedge dy \wedge dz}{x^2+y^2+z^2} - 2\frac{dx \wedge dy \wedge dz}{x^2+y^2+z^2} \\ &= \frac{dx \wedge dy \wedge dz}{x^2+y^2+z^2} \end{aligned}$$

and to take the integral we use polar co-ordinates and since the angles don't matter the integral will be the volume of the sphere

$$\int_{S^2} d\theta d\phi = \int_1^2 \frac{1}{r^2} \cdot r^2 dr = 4\pi$$

so we have that $\int_M d\omega = \int_{\partial M} \omega$ as desired.

**Question 11:**

**Question 12:** (Hitchin page 61)

*Brouwer's fixed point theorem:* Let $B$ be the unit ball $\{x \in \mathbb{R}^n : ||x|| \leq 1\}$ and let $F : B \to B$ be a smooth map from $B$ to itself. Then $F$ has a fixed point, that is, there exists $x$ such that $F(x) = x$.

*proof:*

**Step 1:** Show that there does not exist a differentiable map $f : B^n \to S^{n-1} = \partial B^n$ such that $f|_{\partial B^n} = identity$. Since $\partial B^n$ is a compact, orientalbe manifold there is a nowhere vanishing $n-1$ form $\omega$ on $S^{n-1}$. If we assume such an $f$ exists then $d(f^*(\omega)) = f(d(\omega)) = 0$ since $\omega$ is an $n-1$ form and $dim(S^{n-1}) = n-1$, so $d\omega = 0$. Now by Stokes Theory we have:

$$0 = \int_B^n d(f^*(\omega)) = \int_{\partial B^n} f^*(\omega) = \int_{\partial B^n} \omega \neq 0$$

since $\omega$ is increasing. This serves as a contradiction.

**Step 2:** Assume there exists a differentiable $g : B^n \to B^n$ with $g(p) \neq p$ for all $p \in B^n$. Then the line through $p$ and $g(p)$ intersects $\partial B^n$ at two points, so the ray starting at $g(p)$ and passing through $p$ intersects $\partial B^n$ at one point.

Let $h : B^n \to \partial B^n$ be the map given by $p \mapsto t_p(p - g(p))$ where $t_p$ is some non-negative real number so that $||g(p) + t_p(p - g(p))|| = 1$.

Then $h$ is a continuous differentiable map and well defined because $g(p) \neq p$ and $h|_{\partial B^n} = identity$ which contradicts Step 1.

So it must be the case that the the Brouwer's fixed point theorem is correct.

**Question 13:** (Clara)

Given $\sigma(u, v) = (a \cos u \cos v, b \cos u \sin v, c \sin u)$, the parameterization of the ellipsoid, the calculation for the Gaussian curvature is as follows:

$$\sigma_u = (-a\sin u\cos v, -b\sin u sin v, c\cos u) \qquad \sigma_v = (-a\cos u\sin v, b\cos u cos v, 0)$$
$$\sigma_{uu} = (= a\cos u\cos v, -b\cos u\sin v, -c\sin u) \qquad \sigma_{uv} = (a\sin u\sin v, -b\sin u\cos v, 0)$$
$$\sigma_{vv} = (-a\cos u\cos v, -b\cos u\sin v, 0)$$

$$n = \frac{\phi_u \times \phi_v}{||\phi_u \times \phi_v||}$$
$$E =< \sigma_u, \sigma_u >= \sin^2 u(a^2\cos^2 v + b^2\sin^2 v) + c^2\cos^2 u$$
$$F =< \sigma_u, \sigma_v >= (a^2 - b^2)\cos u\cos v\sin u\sin v$$
$$G =< \sigma_v, \sigma_v >= \cos^2 u[(a^2 - b^2)\sin^2 v + b^2]$$
$$L = \phi_{uu} \cdot n = \frac{abc\cos u}{\sqrt{b^2c^2\cos^4 u\cos^2 v + a^2c^2\cos^4 u\sin^2 v + a^2b^2\cos^2 u\sin^2 u}}$$
$$M = \phi_{uv} \cdot n = 0$$
$$N = \phi_{vv} \cdot n = \frac{abc\cos^3 u}{\sqrt{b^2c^2\cos^4 u\cos^2 v + a^2c^2\cos^4 u\sin^2 v + a^2b^2\cos^2 u\sin^2 u}}$$
$$K = \frac{LN - M^2}{EG - F^2} = \text{gaussian curvature}$$

**Question 14:**

**Question 15:**