Evariste Galois

Galois theory is one of the most fascinating and enjoyable branches of algebra. The problems with which it is concerned have a long and distinguished history. The problems of duplicating a cube or trisecting an angle go back to the Greeks, and the problem of solving cubic, quartic or quintic equations to the Renaissance. But before we delve into the theory that Galois put forth it is important (and enjoyable) to learn about his eccentric life.

Galois' History

Galois was born in the town of Bourg-la-Reine, on the outskirts of Paris, on October 25, 1811. Up to the age of twelve, he was educated entirely by his mother, who instilled in him a knowledge of the classics and a skeptical attitude towards religion. In 1823 the future mathematician began his formal education by entering the prestigious Lycee Louis-le-Grand private school. Almost as soon as he arrived, there had been a revolt by a number of the students who suspected that the school was reinstating conservative Jesuits to staff. The outcome was that forty boys were expelled.

At first, Galois did well in examinations, coming out near the top in the Concours General (general curriculum), although weak at rhetoric. In his fourth year, Galois made much progress studying mathematics at the expense of all his other subjects. He believed that he was ready to enter the Ecole Polytechnique, but due to his lack of preparation in the standard syllabus, failed the difficult entrance examination. It is unofficially recorded that Galois actually failed because he refused to answer questions that he deemed trivial in nature, reportedly throwing his eraser in frustration at the examiner.

Figure 1: Evariste Galois

Galois' mathematical career was also wrought with misfortune, in addition to having most of his work ignored, it was completely misplaced by caretakers on several occasions. When Galois gave

Caucy a paper containing his most important results to present (without keeping a copy himself), Caucy proceeded to lose it. When Galois submitted a paper for the Academie's prize in math, Fourier took the paper home to peruse, but died shorty thereafter and this paper was also lost. Poisson returned a second paper containing the most important results to Galois while he was imprisoned for threatening the life of the king at the time (he toasted to the king by raising a dagger instead of a glass), thus limiting Galois' ability to retort.

On May 30 1828, Galois fought a duel with pistols. There is some doubt about the identity of his opponent, but it seems to have been a prominent Republican activist named Pescheux d'Herbinville. However, the outcome was that Galois was shot in the abdomen and lay unattended for hours until a passer-by took him to the Cochin hospital.

Interestingly enough, Galois was convinced that he was going to die over something small and contemptible and the tragedy is that he let it happen. It would seem that a love affair had something to do with it, but there are indications that he already believed he was doomed much earlier. Quoting a letter Galois sent from prison, he says:

> *'And I tell you that I will die in a duel over some low-class coquette. Why? Because she will invite me to avenge her honour which another has compromised'*

Various imaginative demonstrations have been built up around these strange words. One recent theory, which seems more plausible than most, is that Galois, wishing to be a martyr to the Republican cause, had arranged a false duel in which he was meant to be killed, with the idea that this would be blamed on the police, and there would then be a riot in protest. As illogical as this sounds, Galois was reasonable enough to document all of his work before his dual with instructions to hand it off to Riemann.

Riemann did a lot of work of interpreting Galois' algebraic theories. As for the transcendental functions, Hermite successfully completed one of Galois' investigations in solving the quintic equations by means of elliptic modular functions, and Jordan brought out the group theory governing the behavior of such functions. In any case, Galois' death is considered a tragedy by most mathematicians who had to waste a lot of time interpreting his work.

Galois Theory

Galois theory is largely concerned with properties of groups of automorphisms of a field. If L is a field, we denote $Aut(L)$ the set of all automorphisms of L. $Aut(L)$ is a group under the usual law of composition.

Supposing that A is a subset of $Aut(L)$. We set

$$\phi(A) = \{k \in L | \sigma(k) = k, \forall \sigma \in A\}$$

It is easy to see that $\phi(A)$ is a subfield of L, which we call the *fixed field* of A. In this way, starting from A we obtain an extension $L : \phi(A)$.

Conversely suppose that that $L : K$ is an extension. We denote by $\Gamma(L : K)$ the set of those automorphisms of L which fix K:

$$\Gamma(L : K) = \{\sigma \in Aut(L) | \sigma(K) = k, \forall k \in K\}$$

When there is no doubt what the larger field L is, we shall write $\gamma(K)$ for $\Gamma(L : K)$.

The operations $A \to \phi(A)$ and $L : K \to \gamma(K)$ establishes a polarity between sets of automorphisms of L and extensions $L : K$. The fundamental theorem of Galois theory describes this polarity in great detail.

Theorem 11.8 Suppose that $L :$ is finite. Let $G = \Gamma(L : K)$, and let $k_0 = \phi(G)$. If $L : M : K_0$, let $\gamma(M) = \Gamma(L, M)$.

- The map ϕ is a one-to-one map from the set of subgroups of G onto the set of fields M intermediate between L and K_0. Where γ is the inverse map.
- A subgroup H of G is normal if and only if $\phi(H) : K_0$ is a normal extension.
- Suppose that $H \lhd G$. If $\sigma \in G$, $\sigma|_{\phi(H)} \in \Gamma(\phi(H), K_0)$. The map $\sigma \to \sigma|_{\phi(H)}$ is a homomorphism of G onto $\Gamma(\phi(H), K_0)$, with kernel H. Thus $\Gamma(\phi(H) : K_0) \cong G/H$.

Proof is omitted.