Braid Words and the Braid Word Equivalency Class

Paul Vrbik

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A braid is constructed by intertwining some number of strings that are attached to top and bottom "bars" such that each string is strictly moving downward. That is, if you were to regard each string as the path of some particle, this particle's distance from the bar of origin would be strictly increasing. For any given braid there is a symbol known as the braid word that uniquely identifies it. In particular, an n-braid can be constructed by iteratively applying the $\sigma_i (i = 1, ..., n-1)$ operator as illustrated in Figure 1.

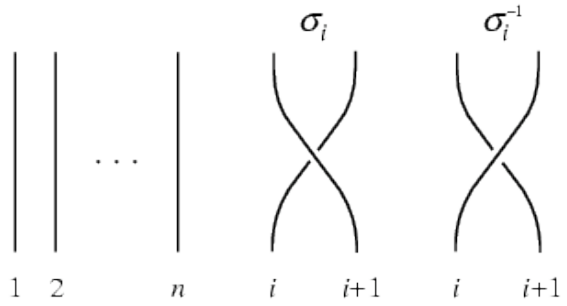

Figure 1: The σ_i operation

This operator switches the lower endpoints of the ith and $(i+1)$th strings – keeping the upper endpoints fixed – with the $(i+1)$th string brought above the ith string. If the $(i+1)$th string passes below the ith string, it is denoted σ_i^{-1}. If we were to list the σ_i's from left to right and then top to bottom (as if we were reading a page of english) this combination of the σ_i and σ_i^{-1} symbols would constitutes a braid word. For example $\sigma_1\sigma_3\sigma_1\sigma_4^{-1}\sigma_2\sigma_4^{-1}\sigma_2\sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_4^{-1}$ is a braid word for the braid illustrated in Figure 2.

Conditions for Topological equivalence for different representations of a braid word are guaranteed by the conditions

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for} |i-j| \geqslant 2 \tag{1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } i \tag{2}$$

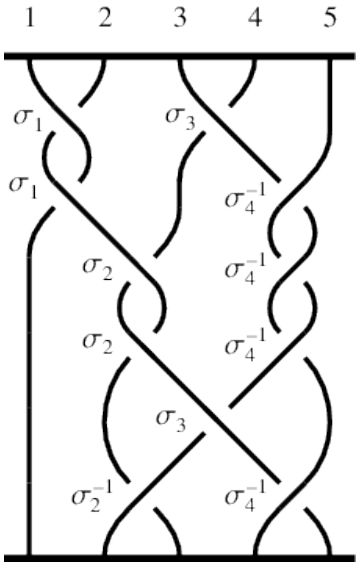

Figure 2: The σ_i operation

The braid relations are taken as the defining relations on the braid group. Each topologically equivalent class of braids represents a collection of words that are different representations for the same braid in the braid group. In principal, the braid relations can be used to show the equivalence of any two words in this collection. Finding a practical solution to word equivalence is called the *word problem*. The word problem is the algebraic analog of the geometric braid equivalence problem. This paper will study novel computational approaches used to solve word equivalence.