Quick Galois Groups

Cubics: (Proposition 22.4)

Let $f(t) = t^3 - s_1 t^2 + s_2 t - s_3 \in \mathbb{Q}[t]$ be irreducible over $\mathbb{Q}$. Then its Galois group is A_3 if

$$D = s_1^2 s_2^2 + 18 s_1 s_2 s_3 - 27 s_3^2 - 4 s_1^3 s_3 - 4 s_2^3$$

is a perfect square in $\mathbb{Q}$, and is S_3 otherwise.

Cubic Resolvent: (mathworld)

For a given a given monic quartic polynomial $f(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ the resolvent cubic is the monic cubic polynomial $g(x) = x^3 + b_2 x^2 + b_1 x + b_0$ where the coefficients b_i are given in terms of the a_i by

$$b_2 = -a_2$$
$$b_1 = a_1 a_3 - 4a_0$$
$$b_0 = 4a_0 a_2 - a_1^2 - a_0 a_3^2$$

The roots β_1, β_2, and β_3 of g are given in terms of the roots $\alpha_1, \alpha_2 . \alpha_3$, and α_4 of f by

$$\beta_1 = \alpha_1 \alpha_2 + \alpha_3 \alpha_4$$
$$\beta_2 = \alpha_1 \alpha_3 + \alpha_2 \alpha_4$$
$$\beta_3 = \alpha_1 \alpha_4 + \alpha_2 \alpha_3$$

Quartics: (From Notes)

Let $f(x) = x^4 + bx^2 + cx + d \in F[x]$ be an irreducible polynomial with $char(F) \neq 2, 3$

$$g(y) = \text{resolvent cubic}$$

$$y_1, y_2, y_4 \text{ roots of } g(y)$$

$G = Galois(E/F)$ where E is a splitting field of $f(x)$ where $D = D(f) = D(g)$ (D is the discriminant).

Then:

$$G \cong S_4 \Leftrightarrow g \text{ irreducible, } D \notin F^2$$
$$G \cong A_4 \Leftrightarrow g \text{ irreducible, } D \in F^2$$
$$G \cong D_4 \Leftrightarrow g \text{ reducible, } D \notin F^2,\ f \text{ is irreducible over } F(\sqrt{D})$$
$$G \cong Z_4 \Leftrightarrow g \text{ reducible, } D \notin F^2,\ f \text{ is reducible over } F(\sqrt{D})$$
$$G \cong V_4 \Leftrightarrow g \text{ reducible, } D \in F^2$$