MATH 2310

Calculus of Science and Engineering

current editor
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THE UNIVERSITY OF NEWCASTLE AUSTRALIA

Contents

1 Functions and Partial Derivatives

This strand provides an introduction to the methods and techniques of the differential and integral calculus for real (scalar) and vector valued functions of several variables.

1.1 Functions of 1 and 2 variables

Definition 1.1 (Function). A *function* is a mapping of elements from one set to another. When f maps elements from A to B we write

$$\begin{aligned} f : A &\to B \\ a &\mapsto b. \end{aligned}$$

It is typical to write $f(a) = b$ when f maps a to b.

Example 1.2. $f(x) = x^2$ is a *function* that maps numbers from $\mathbb{R}$ to numbers in $\mathbb{R}^{>0}$. For instance we can say

$$f \textit{ maps } -3 \textit{ to } 9$$

or just write $f(-3) = 9$.

Definition 1.3 (Domain). Let $f : A \to B$. The *domain* of f is the subset of A for which the function is defined:

$$\operatorname{dom} f := \{a \in A : f(a) \text{ is defined}\}.$$

In practice finding the domain of a function usually boils down to ensuring we never divide by zero or take square-roots of negative numbers. See Figures 1.1 and 1.2.

Definition 1.4 (Range). Let $f : A \to B$. The *range* of f is the subset of B which is "reachable" by f.[a]

$$\operatorname{rng} f := \{f(a) : a \in \operatorname{dom} f\} \subseteq B.$$

[a] Strickly speaking y is "reachable" when $\exists x \in \operatorname{dom}(f) : f(x) = y$.

We are mostly interested in *real valued functions*, that is, functions given by $f : \mathbb{R}^n \to \mathbb{R}$ for some $n \in \mathbb{N}^{>0}$.

1.1.1 Graphs

We visualize real functions by plotting their *graphs*. The graph of f is the collection of points that are said to *satisfy* it. A *plot* is some visual representation of that set. See Figure 1.3.

Definition 1.5 (Univariate Function). When f is the function given by

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto y \end{aligned}$$

the resulting function defines a *real function of one variable*. We call x the *independent variable* and y the *dependent variable* because the value of y *depends* on x.

Note that strictly speaking a function must also satisfy the so-called "vertical line test." Practically speaking this means our equation in the variables x and y can be solved for y. We write $y = f(x)$ to emphasize that this rewriting is possible.

Definition 1.6 (Graph of univariate function). Let $f : \mathbb{R} \to \mathbb{R}$. The *graph* of the *univariate* function f is the collection of points given by

$$\mathcal{G}(f) := \{(x, f(x)) : x \in \operatorname{dom} f\} \subseteq \mathbb{R} \times \mathbb{R}.$$

Example 1.7. Let $f(x) = x^2$, then

$$\mathcal{G}(f) = \{(0,0), (1,1), (-1,1), (2,4), (-2,4), \ldots\}.$$

Note the "..." here are somewhat misleading as there is no way to enumerate the points of $\mathbb{R} \times \mathbb{R}$.

1.1.2 Surfaces

Functions with *two independent* variables define *surface*.

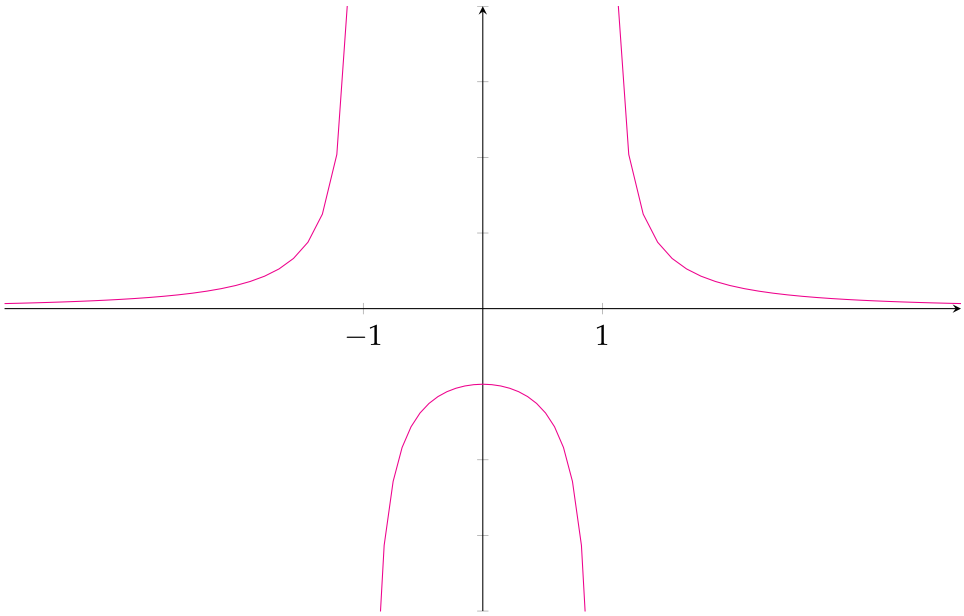


Figure 1.1: The function $f(x) = \dfrac{1}{x^2 - 1}$ is plotted and

$$\operatorname{dom} f = \mathbb{R} - \{-1, 1\}, \qquad \operatorname{rng} f = \mathbb{R}.$$

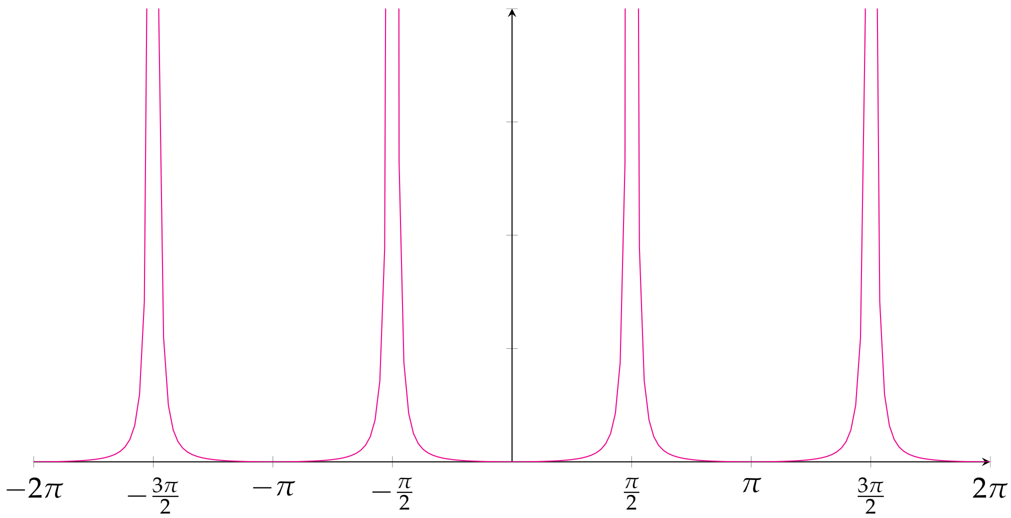


Figure 1.2: The function $f(x) = \tan^2 x = (\tan x)^2$ is plotted and

$$\operatorname{dom} f = \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z}^{\neq 0} \right\}, \qquad \operatorname{rng} f = \mathbb{R}^{\geq 0}.$$

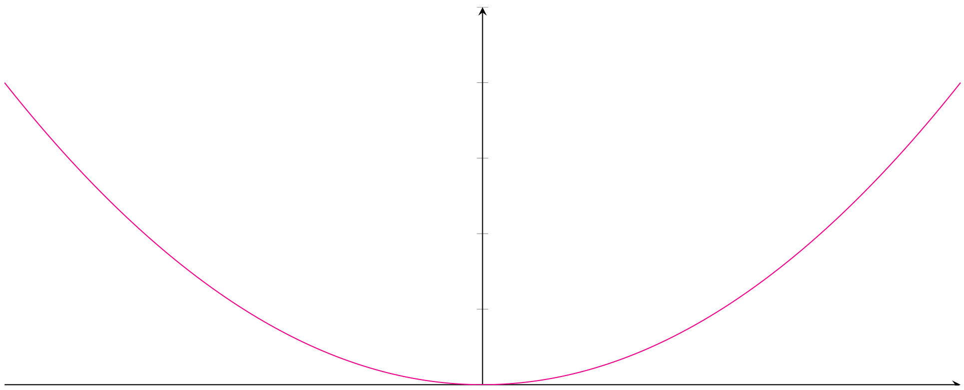


Figure 1.3: The *graph* of a parabola plotted.

Definition 1.8 (Bivariate Function). When f is given by

$$\begin{aligned} f : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto z \end{aligned}$$

the resulting function defines a *real function of two variables*. Here x and y are the *independent variables* and z is the *dependent variable*.

As we are mapping *ordered pairs* (x, y) to z it would be proper to write $f((x, y)) = z$. We write $f(x, y) = z$ instead as the double brackets are ugly and unnecessary. Similarly, when summing points (or vectors), we write $f((x, y) + (a, b))$ rather than $f(x + a, y + b)$.

Notice our definition of *domain* and *range* do not require modification despite the addition of another independent variable. For $f : \mathbb{R}^2 \to \mathbb{R}$ we need only set $A = \mathbb{R}^2$ and $B = \mathbb{R}$ in the definitions to obtain:

$$\begin{aligned} \operatorname{dom} f &= \{(x, y) \in \mathbb{R}^2 : f(x, y) \text{ is defined}\} \\ \operatorname{rng} f &= \{f(x, y) : (x, y) \in \operatorname{dom} f\} \subseteq \mathbb{R}. \end{aligned}$$

Definition 1.9 (Graph of bivariate function). Let $f : \mathbb{R}^2 \to \mathbb{R}$. Then the *graph* of the *bivariate* real function f is the collection of points given by

$$\mathcal{G}(f) := \{(x, y, f(x, y)) : (x, y) \in \operatorname{dom} f\} \subseteq \mathbb{R}^3.$$

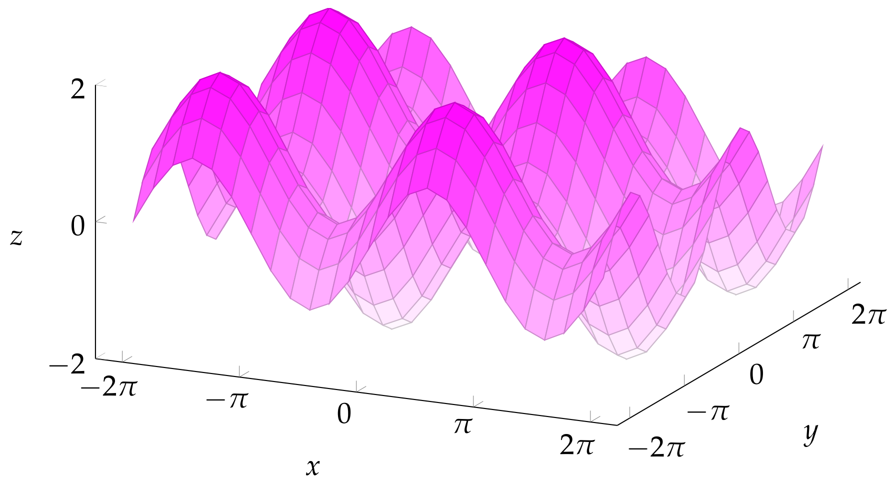


Figure 1.4: The function $f(x, y) = \sin x + \sin y$ is plotted and

$$\operatorname{dom} f = \mathbb{R}, \qquad \operatorname{rng} f = [-2, 2].$$

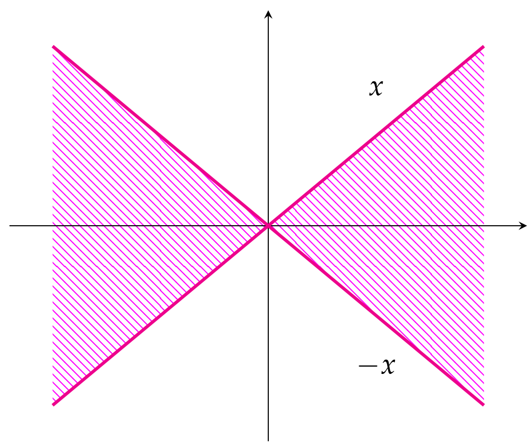


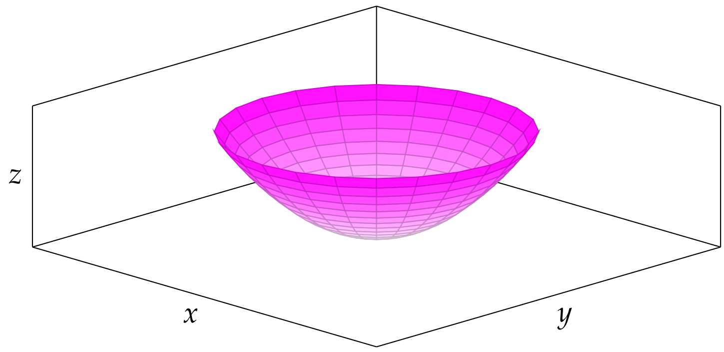
Figure 1.5: The function $f(x, y) = \sqrt{x^2 - y^2}$ has

$$\operatorname{dom} f = \{(x, y) : (x + y)(x - y) \geq 0\}$$

as illustrated above.

1.2 Contour Maps Slopes and Gradients

Consider the *paraboloid* given by the function $f(x,y) = x^2 + y^2$:

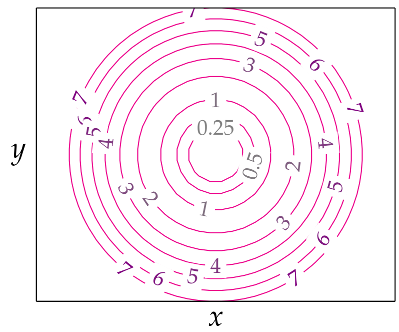


After "cutting" this paraboloid $z = x^2 + y^2$ by the plane $z = 2$ we are left with the circle $2 = x^2 + y^2$. These *equations* that remain after cutting a bivariate function by some plane $z = a : a \in \mathbb{R}$ define a family of curves called *level curves*.

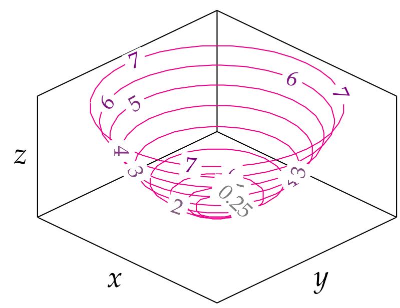
Definition 1.10 (Level Curves). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bivariate function. The *level curves* of f are given by

$$\mathcal{L}(f) := \{f(x,y) = a : a \in \mathbb{R}\}.$$

Definition 1.11 (Contour Map). A *contour map* of f is a plot that illustrates some of the level curves of f.



Level curves of the paraboloid.



Contour plot of paraboloid, isometric view.

Note the numbers indicate the "level" at which the cut was made. Namely, they indicate what value z has been set to.

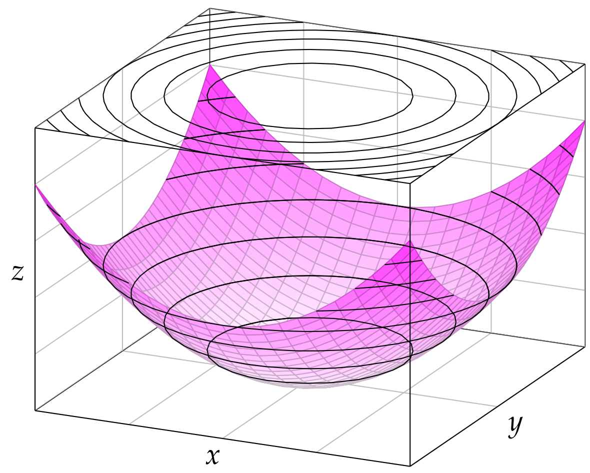


Figure 1.6:
Paraboloid $x^2 + y^2$.

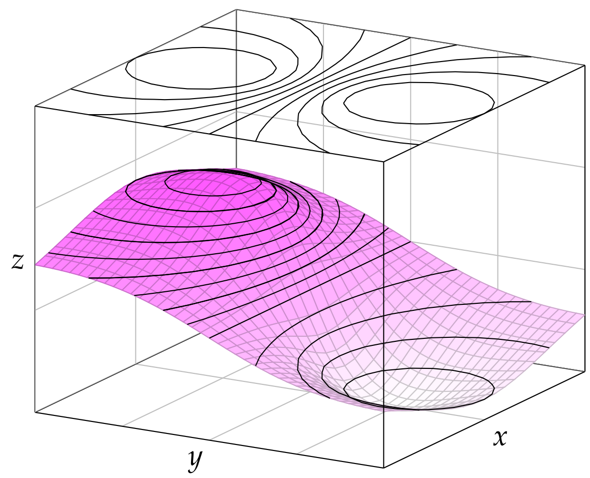


Figure 1.7:
$f(x, y) = \dfrac{y}{x^2 + y^2 + 1}$.

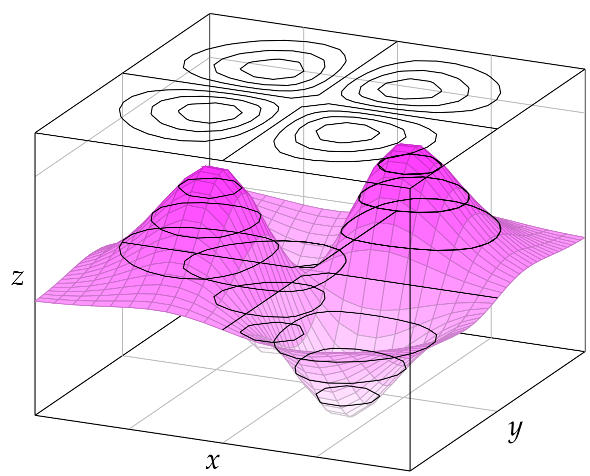


Figure 1.8:
$f(x, y) = xye^{-x^2-y^2}$.

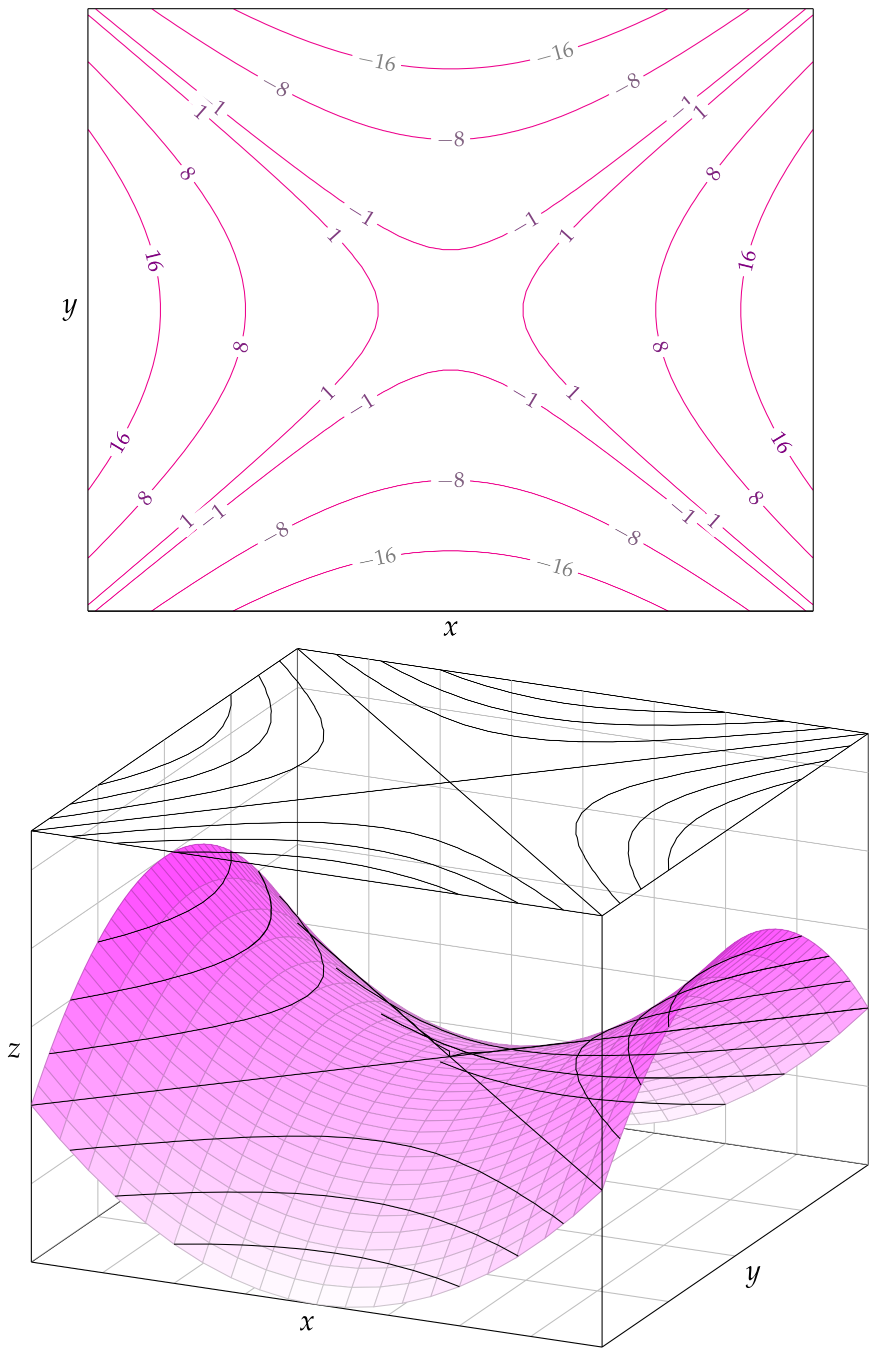
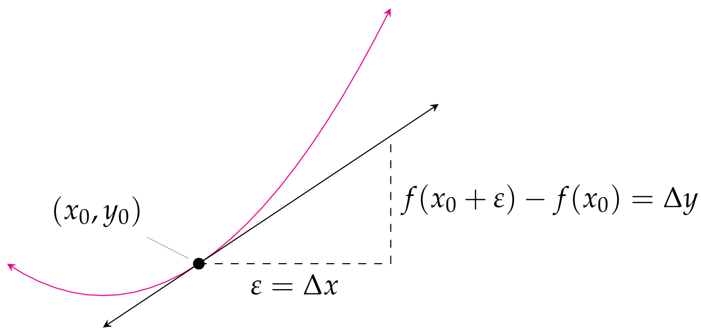


Figure 1.9: The curve $f(x, y) = x^2 - y^2$ and its contour lines.

1.3 Slopes and Gradients

1.3.1 Univariate

Recall the slope of a tangent line is found by calculating the "instantaneous rise over run". This is done by taking $\varepsilon \to 0$ in the configuration below:



Definition 1.12 (Tangent Line). Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x_0) = y_0$. The *tangent line* of f at (x_0, y_0) is

$$(y - y_0) = m(x - x_0)$$

where $m = f'(x_0)$ is the *instantaneous slope* of f at (x_0, y_0) given by

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x_0) := \lim_{\varepsilon \to 0} \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon}.$$

The *instantaneous slope* is a measure of how fast a function is increasing (or decreasing) at a given point — see Figure 1.10.

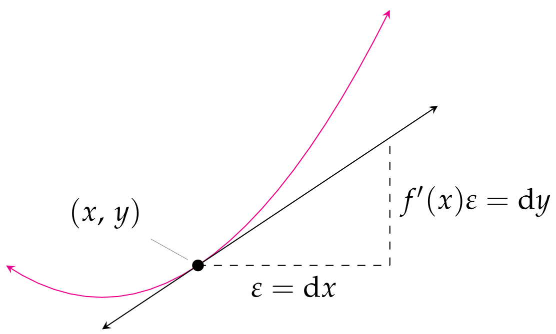
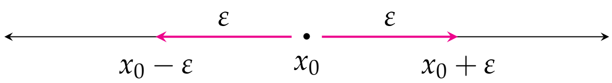


Figure 1.10: Infinitesimally small changes on $f(x)$. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the "rise over run" or "instantaneous slope".

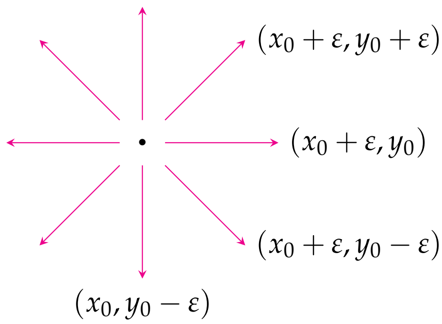
1.3.2 Bivariate

In the *univariate* case the slope tells us how fast (or slow) to vary the *dependent* variable as the *independent* variable is increased by infinitesimal distances. In the *bivariate* case we need the slope to tell us how to vary the *dependent* variable as the *two independent* variables are move by infinitesimal distances.

Our first concern is that of 'moves' in $\mathbb{R}^2$. There are only two ways to move away from x_0 in $\mathbb{R}$:



But an infinite number of ways to move away from (x_0, y_0) in $\mathbb{R}^2$:

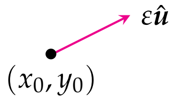


We rectify this by specifying a direction for the derivative. (In the univariate case the direction was assumed to be forward.)

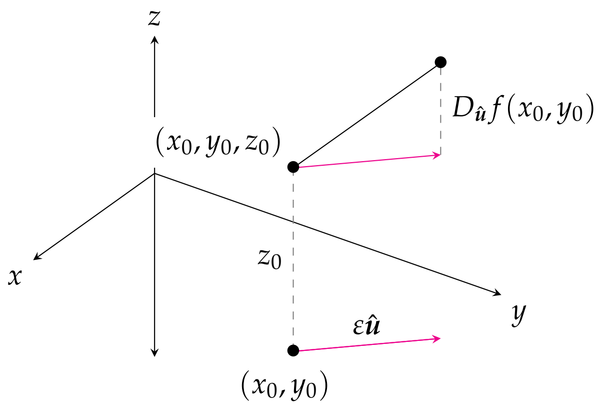
Definition 1.13 (Directional Derivative)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in \operatorname{dom} f$. The *directional derivative* in the unit direction $\hat{\boldsymbol{u}}$ is

$$D_{\hat{\boldsymbol{u}}} f(x_0, y_0) := \lim_{\varepsilon \to 0} \frac{f((x_0, y_0) + \varepsilon \hat{\boldsymbol{u}}) - f(x_0, y_0)}{\varepsilon}.$$

Now, if we move ε units in the direction of $\hat{\boldsymbol{u}}$ from (x_0, y_0) as in



then z increases by $D_{\hat{\mathbf{u}}} f(x_0, y_0)$:



1.4 Partial Derivatives

From single variable calculus we know that limits for finding instantaneous slopes can be bypassed because

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}.$$

That is, f' can be determined *algebraically* by using rules of derivatives instead of evaluating a limit. Let us develop an algebraic way to determine $D_{\hat{\mathbf{u}}} f$ for multivariate functions as we would like to avoid taking limits here as well.

Example 1.14. Let $f(x, y) = x^3 + x^2 y^3 - 2y^2$ and $a \in \mathbb{R}$ a *constant*. Then

$$f(x, 2) = x^3 + x^2 \cdot 2^3 - 2 \cdot 2^2 = x^3 + 8x^2 - 8$$

is a *univariate* function whose derivative we calculate in the usual way:

$$\frac{\mathrm{d} f(x, 2)}{\mathrm{d}x} = 3x^2 + 16x - 8.$$

Geometrically we are taking the derivative of the *univariate* function that is left when cutting a surface with a plane.

The derivative obtained by setting all but one independent variable to constants before deriving is called the *partial derivative*.

Partial Derivative. Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a bivariate function. The *partial derivative of f with respect to x* at (a, b) is given by

$$\frac{\partial f}{\partial x}(a, b) := \frac{\mathrm{d} f(x, b)}{\mathrm{d}x}(a).$$

And similarly, the *partial derivative of f with respect to y* at (a,b) is given by

$$\frac{\partial f}{\partial y}(a,b) := \frac{\mathrm{d}f(a,y)}{\mathrm{d}y}(b).$$

Notation. If $z = f(x,y)$ then the following are equivalent notations for partial derivatives

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x,y) = \frac{\partial z}{\partial x} = D_x f,$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f(x,y) = \frac{\partial z}{\partial y} = D_y f.$$

Rules for finding partial derivatives. Let $z = f(x,y)$. Then

1. To find $\frac{\partial f}{\partial x}$ regard y as constant and differentiate $f(x,y)$ with respect to x.
2. To find $\frac{\partial f}{\partial y}$ regard x as constant and differentiate $f(x,y)$ with respect to y.

Question 1.15. Let $f(x,y) = x^3 + x^2y^3 - 2y^2$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

ANSWER. $\dfrac{\partial f}{\partial x} = 3x^2 + 2xy^3 - 0$ and $\dfrac{\partial f}{\partial y} = 0 + 3x^2y^2 - 4y$. ◆

Question 1.16. Let $f(x,y) = \sin\left(\frac{x}{1+y}\right)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

ANSWER. Here chain rule is needed:

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{-x}{(1+y)^2}\ .$$

◆

Question 1.17. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when z is *implicitly defined* by $x^3 + y^3 + z^3 + 6xyz = 1$.

ANSWER. First we partially differentiate with respect to x remembering that z is a function of x:

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0.$$

Then we solve for $\frac{\partial z}{\partial x}$ to obtain

$$\frac{\partial z}{\partial x} = \frac{x^2 + 2yz}{z^2 + 2xy}.$$

The remaining derivative is left as an exercise. ◆

Definition 1.18 (Higher Order Partial Derivatives). Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$. The *nth partial derivative of f with respect to x* is given by

$$\frac{\partial^n f}{\partial x^n} := \frac{\partial}{\partial x} \frac{\partial^{n-1} f}{\partial x^{n-1}}$$

that is

$$\frac{\partial^n f}{\partial x^n} := \underbrace{\frac{\partial}{\partial x} \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x}}_{n \text{ times}} f.$$

The definition for y is analogous.

Question 1.19. Let $f(x, y) = \sin(3x + y)$. What are the first four derivatives with respect to x of f?

ANSWER.

$$\frac{\partial f}{\partial x} = 3\cos(3x + y) \qquad \frac{\partial^2 f}{\partial x^2} = -3^2 \sin(3x + y)$$

$$\frac{\partial^3 f}{\partial x^3} = -3^3 \cos(3x + y) \qquad \frac{\partial^4 f}{\partial x^4} = 3^4 \sin(3x + y)$$

◆

Exercise 1.1. Find the nth partial derivative with respect to x of $f(x, y) = \sin(3x + y)$.

1.4.1 Chain Rule

Recall the *chain rule* for univariate functions.

Chain Rule. If $y = f(x)$ and $x = g(t)$ where f and g are differentiable functions, then y is indirectly a differentiable function of t and in particular:

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}.$$

Equivalently $\frac{df(g(x))}{dx} = f'(g(x)) \cdot g'(x)$.

For bivariate functions, we have two types of chain rules.

Chain Rule – Type 1. Suppose $z = f(x, y)$ with $x = g(t)$ and $y = h(t)$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Chain Rule – Type 2. Suppose $z = f(x, y)$ and $x = g(s, t)$ and $y = h(s, t)$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Question 1.20. Let $z = x^2y + 3xy^4$ with $x = \sin 2t$ and $y = \cos t$. Find $\frac{\mathrm{d}z}{\mathrm{d}t}$.

ANSWER. Using chain rule

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)\end{aligned}$$

◆

Question 1.21. Let $z = \mathrm{e}^x \sin y$, $x = st^2$ and $y = s^2t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

ANSWER. Applying Type 2 chain rule we obtain

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ &= (\mathrm{e}^x \sin y)\,(t^2) + (\mathrm{e}^x \cos y)\,(2st)\\ &= t^2\mathrm{e}^{st^2}\sin(s^2t) + 2st\mathrm{e}^{st^2}\cos(s^2t).\end{aligned}$$

$\frac{\partial z}{\partial t}$ is left as an exercise. ◆

1.4.2 Gradient

Let us return now to the *directional derivative*. If we let $g(h) = f((x_0, y_0) + h\hat{\boldsymbol{u}})$ in Definition 1.13 then the directional derivatives writes

$$D_{\hat{\boldsymbol{u}}} f(x_0, y_0) = \lim_{\varepsilon\to 0} \frac{g(0+\varepsilon) - g(0)}{\varepsilon} = g'(0).$$

On the other hand, if we let $(x, y) = (x_0, y_0) + h\boldsymbol{u}$ so that $x = x_0 + hu$ and $y = y_0 + hv$ (i.e. $x \to x_0$ and $y \to y_0$ when $h \to 0$) then

$$g(h) = f((x_0, y_0) + h\boldsymbol{u}) = f(x, y)$$

and by chain rule we have

$$g'(h) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} = \frac{\partial f}{\partial x}\cdot u + \frac{\partial f}{\partial y}\cdot v$$

and thereby

$$g'(0) = \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \boldsymbol{u}.$$

Definition 1.22 (Gradient)**.** The *gradient* of a bivariate function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is given by *the vector*

$$\nabla f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

Proposition 1.23. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function then:

$$D_{\hat{\boldsymbol{u}}} f(x, y) = \nabla f \cdot \boldsymbol{u}.$$

Thus we have accomplished our goal of bypassing limits to determine 'slope' in $\mathbb{R}^3$.

Question 1.24. What is the gradient of $f = x^2y + y^2$?

ANSWER. $\nabla f = \langle 2xy, x^2 + 2y \rangle$. ◆

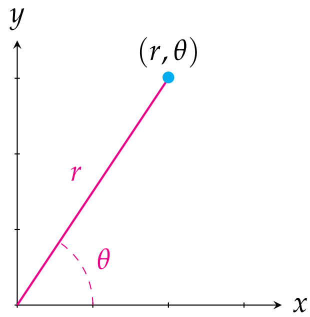
2 Coordinate Systems

In this chapter we investigate various ways points in space can be encoded. We do this because some problems are easier to solve after changing the way points are referenced.

2.1 Polar Coordinates

Points in polar are given by a *length* and *angle*. We insist that all angles remove "extra" 2π's and when this is the case θ is called the *principle argument*. For example $\frac{7}{2}\pi = \left(2 + \frac{3}{2}\right)\pi \equiv \frac{3}{2}\pi$.

Definition 2.1 (Polar Coordinates). The set of Polar coordinates is given by



$$\mathbb{R} \times [0, 2\pi) = \{(r, \theta) : r \in \mathbb{R} \wedge \theta \in [0, 2\pi)\}.$$

2.1.1 Polar-Cartesian Conversion

Conversion between polar and Cartesian coordinates is easy — we need only some basic trig. See Figure 2.3.

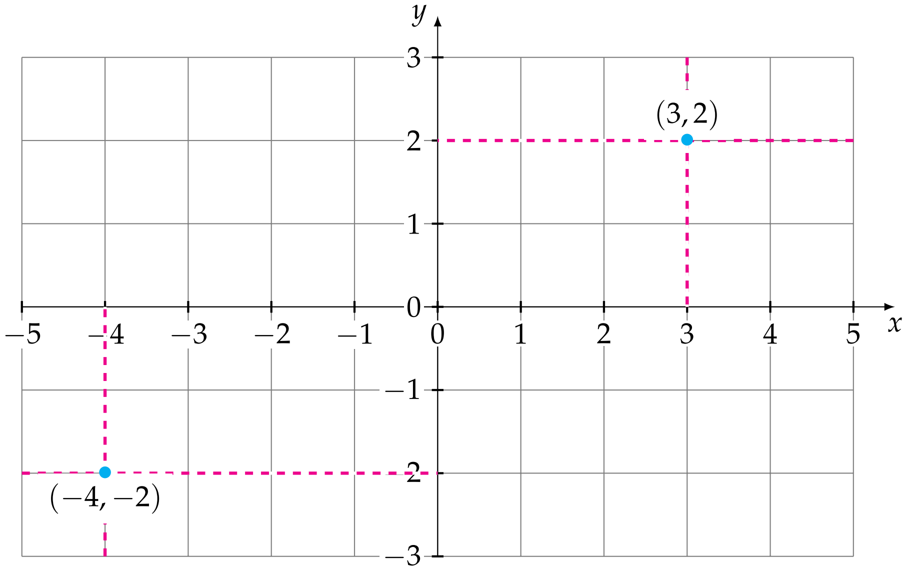


Figure 2.1: Points of the *Cartesian* or *rectangular* coordinate system.

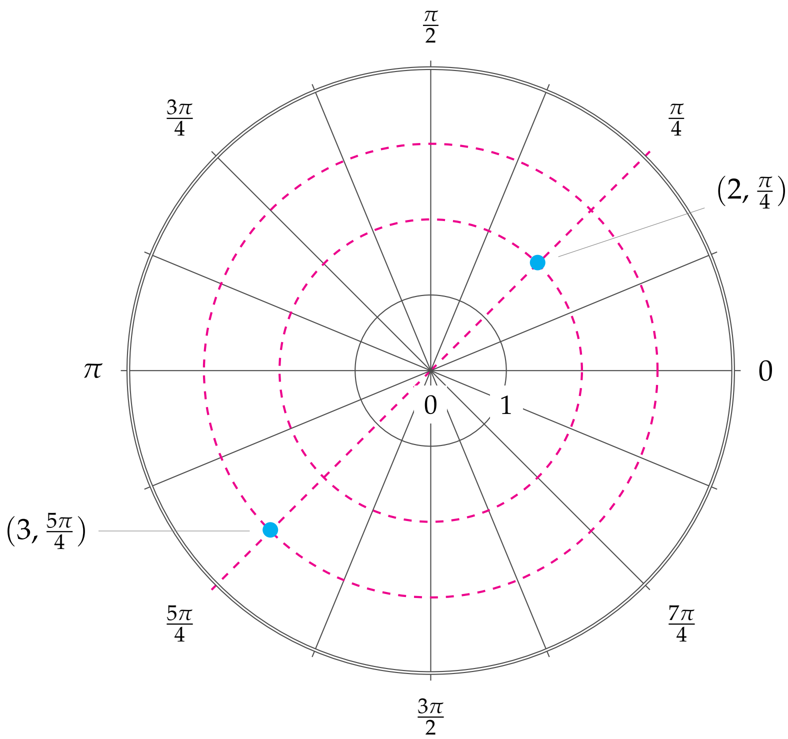


Figure 2.2: Points of the *polar* coordinate system.

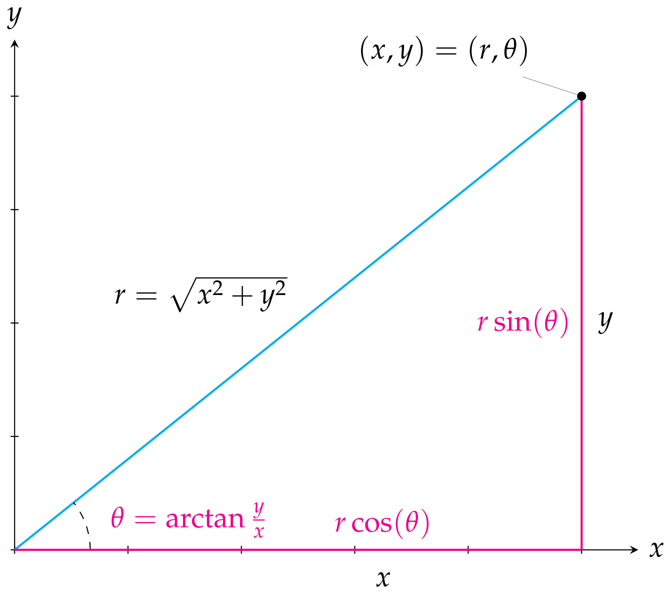


Figure 2.3: Cartesian-Polar conversion.

Cartesian to Polar.

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R} \times [0, 2\pi) \\ (x, y) &\mapsto \left(\sqrt{x^2 + y^2},\, \arctan \frac{y}{x} \right). \end{aligned} \tag{2.1}$$

Polar to Cartesian.

$$\begin{aligned} \mathbb{R} \times [0, 2\pi) &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\cos\theta,\, r\sin\theta)\,. \end{aligned} \tag{2.2}$$

Question 2.2. What is the equation for the *unit circle* on the *polar* coordinate grid?

ANSWER. The unit circle is comprised of those points which are unit distance from the origin. In polar, these points satisfy *r=1*. ◆

Question 2.3. The equation for the unit circle in polar is $r = 1$. What is the equation of the unit circle in Cartesian?

ANSWER. Applying (2.1) to the equation $r = 1$ gives

$$r = 1 \implies \sqrt{x^2 + y^2} = 1 \implies x^2 + y^2 = 1^2.$$

◆

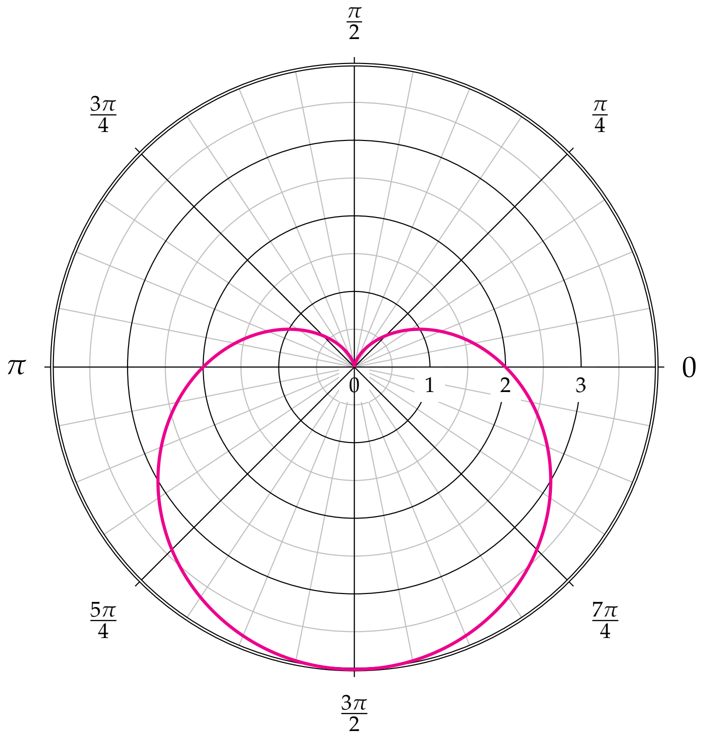


Figure 2.4: The graph of $r = 2 - 2\sin\theta$.

Question 2.4. What is the *Cartesian* equation for the *polar* equation $r = 2 - 2\sin\theta$?

ANSWER. Applying 2.2 gives

$$r = 2 - 2\sin\theta \implies \sqrt{x^2+y^2} = 2 - 2\sin\arctan\frac{y}{x}$$
$$\implies \sqrt{x^2+y^2} = 2 - \frac{2y}{\sqrt{x^2+y^2}} \implies x^2+y^2 = 2\sqrt{x^2+y^2} - 2y.$$

◆

Question 2.5. Convert the function $2x - 5x^3 = 1 + xy$ to *polar*.

ANSWER. Apply the transformation $(x, y) \leftarrow (r\cos\theta, r\sin\theta)$.

$$2x - 5x^3 = 1 + xy \implies 2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta.$$

◆

Question 2.6. Convert the *polar* equation $r = -8\cos\theta$ to *Cartesean*.

ANSWER. As there is no direct substitution for $\cos\theta$ this seems harder. However, notice if we multiply the equation by r we get:

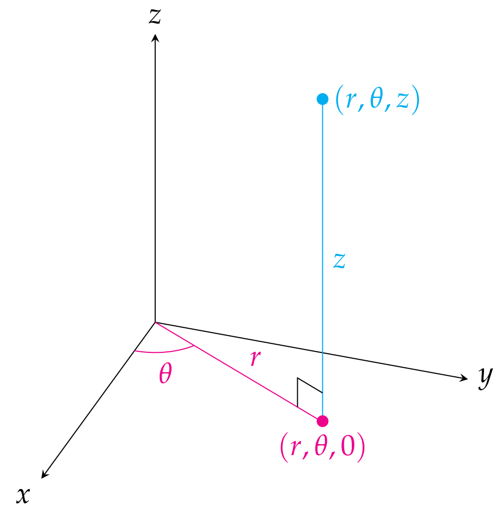
$$r = -8\cos\theta \implies r^2 = -8r\cos\theta \implies x^2+y^2 = -8x.$$

◆

2.2 Cylindrical Coordinates

Cylindrical coordinates are basically *polar coordinates* plus *height*. They are used when there is symmetry about a line, taken to be the z-axis. In cylindrical coordinates this cylinder has the simple equation $r = c$.

Definition 2.7 (Cylindrical Coordinates). The set of Cylindrical coordinates is given by



$$\mathbb{R} \times [0, 2\pi) \times \mathbb{R} = \{(r, \theta, z) : r \in \mathbb{R} \wedge \theta \in [0, 2\pi) \wedge z \in \mathbb{R}\}.$$

2.2.1 Cylindrical-Cartesian Conversion

We convert from *Cartesian to cylindrical* by using the polar conversions and setting $z = z$.

Cylindrical to Cartesian. Let $r^2 = x^2 + y^2$.

$$\begin{aligned} \mathbb{R} \times [0, 2\pi) \times \mathbb{R} &\to \mathbb{R}^3 \\ (r, \theta, z) &\mapsto (r\cos\theta, r\sin\theta, z). \end{aligned} \tag{2.3}$$

Cartesian to Cylindrical.

$$\begin{aligned} \mathbb{R}^3 &\to \mathbb{R} \times [0, 2\pi) \times \mathbb{R} \\ (x, y, z) &\mapsto \left(\sqrt{x^2 + y^2}, \arctan\frac{y}{x}, z\right). \end{aligned} \tag{2.4}$$

Question 2.8. What is the equation of the unit cylinder (see Figure 2.5) in Cartesian that has $x^2 + y^2 = 1$ as its base?

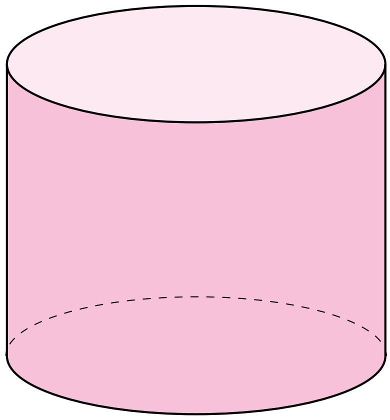


Figure 2.5:
A (truncated) cylinder.

ANSWER. $x^2 + y^2 = 1$. We get this circle for every $z \in \mathbb{R}$. ◆

Question 2.9. What is the equation of the unit cylinder from the previous question in Polar?

ANSWER. $r = 1$. ◆

Question 2.10. Identify the shape given by $r^2 = (10 + z)(10 - z)$.

ANSWER. Let us convert back to Cartesian

$$r^2 = (10+z)(10-z) \implies r^2 = 10^2 - z^2$$
$$\implies \left(\sqrt{x^2+y^2}\right)^2 + z^2 = 10^2 \implies x^2 + y^2 + z^2 = 10^2.$$

This is a *sphere* of radius 10. ◆

Question 2.11. Find the equation of the *cone* in Figure 2.7 in *cylindrical*.

ANSWER. We need to draw a circle for every $z \in [0, h]$ whose radius linearly decreases as z increases. We know that $r = R$ when $z = 0$ and $r = 0$ when $z = h$. Thus (we guess) the equation is

$$r = R - \frac{zR}{h} : z \in [0, h].$$

◆

2.3 Spherical Coordinates

Spherical coordinates are useful when there is a symmetry about a point taken to be the origin. These coordinates are given by a polar coordinate plus *height* given by an *azimuthal angle*. Unsurprisingly, the unit sphere is trivially defined in this coordinate system by $r = 1$.

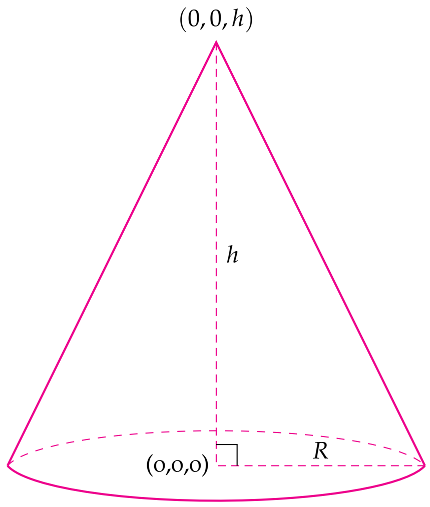
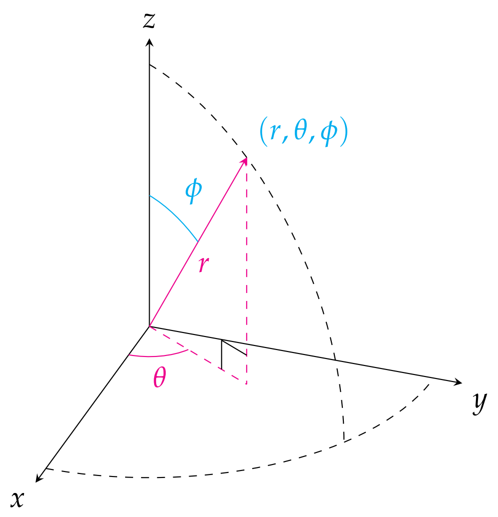


Figure 2.6:
A cone labelled in Cartesian.

Definition 2.12 (Spherical Coordinates). The set of *spherical coordinates* is given by



$$\{(r, \theta, \phi) : r \in \mathbb{R} \wedge \theta \in [0, 2\pi) \wedge \phi \in [0, \pi]\}$$

r = radial distance. θ = polar angle. ϕ = azimuthal angle.

2.3.1 Spherical-Cartesian Conversion

Spherical-Cartesian conversion is performed by using trigonometry. See Figure 2.7.

Spherical to Cartesian.

$$\begin{aligned}\mathbb{R}\times[0,2\pi)\times[0,\pi] &\to \mathbb{R}^3\\ (r,\theta,\phi) &\mapsto (r\sin\phi\cos\theta,\ r\sin\phi\sin\theta,\ r\cos\phi)\,. \end{aligned} \tag{2.5}$$

Cartesian to Spherical.

$$\begin{aligned}\mathbb{R}^3 &\to \mathbb{R}\times[0,2\pi)\times[0,\pi]\\ (x,y,z) &\mapsto \left(\sqrt{x^2+y^2+z^2},\ \arccos\sqrt{x^2+y^2+z^2},\ \arctan\frac{y}{x}\right). \end{aligned} \tag{2.6}$$

Question 2.13. Find the spherical equation of the unit sphere and confirm your answer by converting the equation to Cartesian.

ANSWER. The unit sphere is given by $r = 1$ (all the points unit distance from the origin at any θ and ϕ). To convert to Cartesian we use (2.6)

$$r = 1 \implies \sqrt{x^2+y^2+z^2} = 1 \implies x^2+y^2+z^2 = 1.$$

Which is indeed the equation for the sphere. ◆

Question 2.14. Convert the point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ into spherical coordinates.

ANSWER. Use (2.5)

$$\begin{aligned} x &= r\sin\phi\cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}\\ y &= r\sin\phi\sin\theta = 2\sin\frac{\pi}{3}\sin\pi 4 = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}\\ z &= r\cos\phi = 2\cos\frac{\pi}{3} = 2\frac{1}{2} = 1. \end{aligned}$$

◆

Question 2.15. Convert the Cartesian coordinate $(0, 2\sqrt{3}, -2)$ into spherical.

ANSWER. We have $r = \sqrt{x^2+y^2+z^2} = \sqrt{0+12+4} = 4$, $\cos\phi = \frac{z}{r} = \frac{-2}{4} = -\frac{1}{2} \implies \phi = \frac{2\pi}{3}$, and $\cos\theta = \frac{x}{r\sin\phi} = 0 \implies \theta = \frac{\pi}{2}$. Note $\theta \neq \frac{3\pi}{2}$ because $y = 2\sqrt{3} > 0$ and thereby the spherical point is given by $\left(4, \frac{2\pi}{3}, \frac{2\pi}{3}\right)$. ◆

Question 2.16. Determine the surface given by the spherical equation $\phi = \frac{\pi}{3}$.

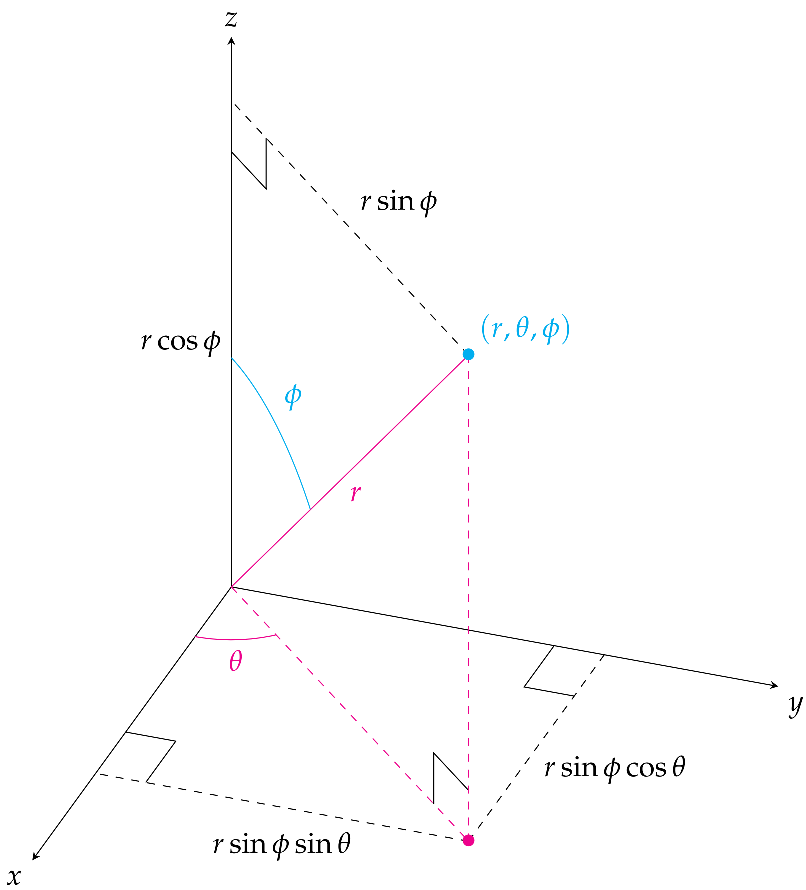


Figure 2.7: Cartesian-Polar conversion.

ANSWER. Independent of how far we move from the origin, or whatever the angle we have rotated about the z-axis, the point must always be at an *azimuthal*-angle of $\frac{\pi}{3}$ — this is a cone. ◆

Question 2.17. What is the surface given by the spherical equation $r \sin \phi = 2$?

ANSWER. Let us square each side of the equation and (as a magic step) add $r^2 \cos^2 \phi$.

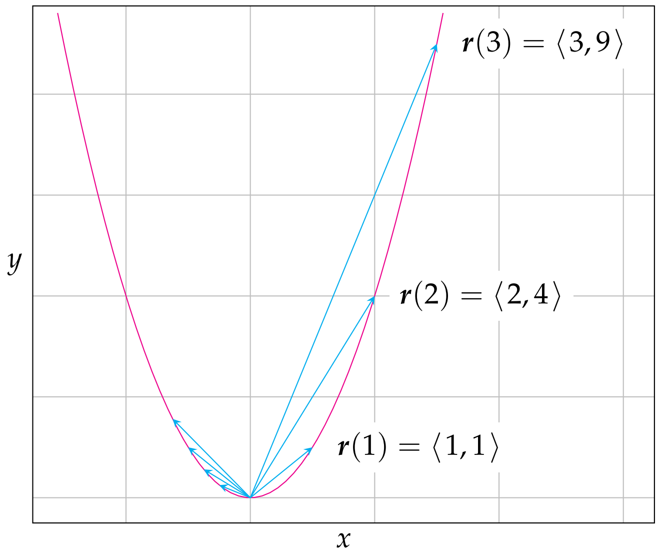
$$
\begin{aligned}
r^2 \sin^2 \phi + r^2 \cos^2 \phi &= 4 + r^2 \cos^2 \phi \\
\implies r^2(\sin^2 \phi + \cos^2 \phi) &= 4 + r^2 \cos^2 \phi \\
\implies r^2 &= 4 + (r \cos \phi)^2 \\
\implies x^2 + y^2 + z^2 &= 4 + z^2 && \text{Converting to Cartesian.} \\
\implies x^2 + y^2 &= 4.
\end{aligned}
$$

Which is a *cone* in $\mathbb{R}^3$. ◆

3 General Curves and Surfaces

3.1 Parametric Curves in $\mathbb{R}^2$

Consider the parabola $y = x^2$. We may regard this curve as a *collection of vectors* generated by the vector valued equation $\boldsymbol{r}(t) = \langle t, t^2 \rangle$ as below:



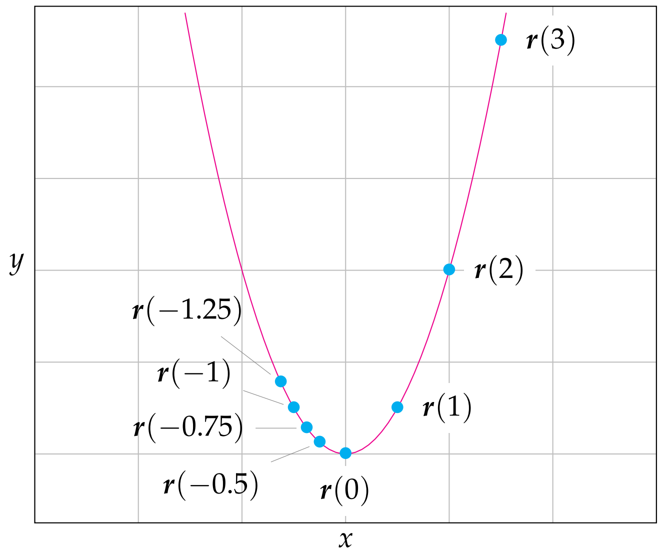
In general this vector-valued function is called the *component function* given by two real-valued *components*.

Component Function 2D. Let $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$. Then

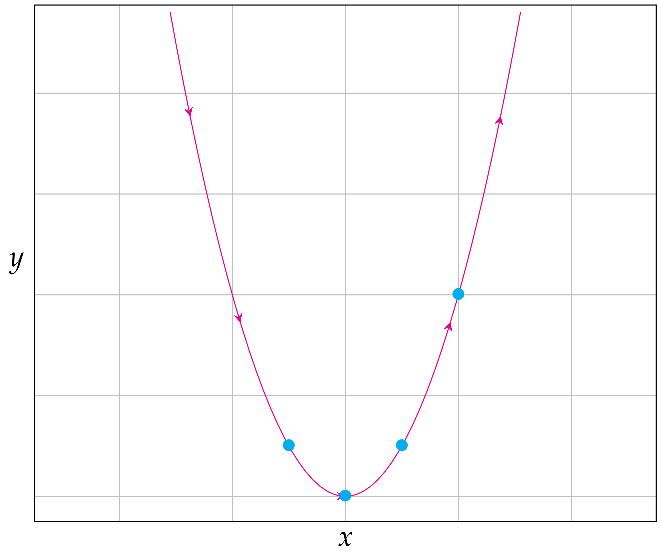
$$\begin{aligned} \boldsymbol{r} : \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto \langle x(t), y(t) \rangle \end{aligned}$$

is the *component function* given by x and y.

The letter t is used as the independent variable because it typically represents time. Namely, when t is *increases* we can regard the component function as encoding *movement* in the plane:



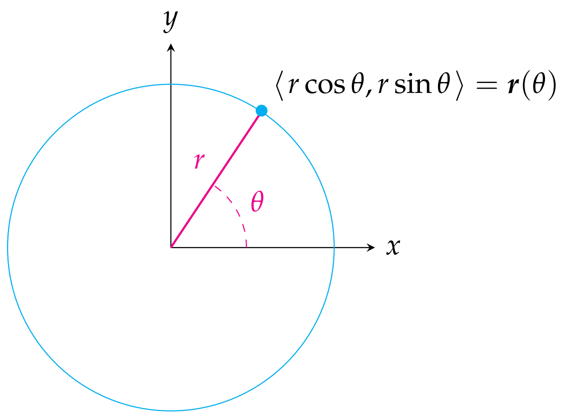
Where we (sometimes) draw arrows on the curve to indicate how the particle moves as time *increases*:



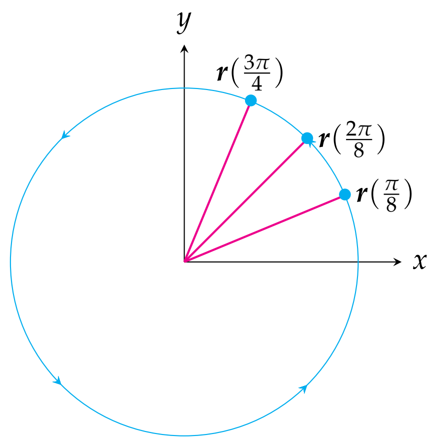
Functions given by component functions are said to be in *parametric form* because points are retrieved through a *parameter* t (though more than one parameter can be used). We can see that every function $y = f(x)$ corresponds to the component function given by $\boldsymbol{r}(t) = \langle t, f(t) \rangle$. However, equations which are *not functions* (like circles) will require extra consideration.

Question 3.1. What is the *parametric form* of the circle of radius r?

ANSWER. The points on the circle can be given by the *single parameter* θ via



and thus the parametric form is $\boldsymbol{r}(\theta) = \langle r\cos\theta, r\sin\theta\rangle$. Moreover, by checking a few values of θ, we can determine which direction the particle is moving. (Remember we have to *increase* θ.)



Counter-clockwise motion. ◆

Question 3.2. Where does the spiral $\boldsymbol{r}(t) = \langle \frac{\cos t}{\sqrt{t}}, \frac{\sin t}{\sqrt{t}}\rangle$ of Figure 3.1 cross the x axis? It will do so infinitely many times.

ANSWER. We need to find all values of t such that $\boldsymbol{r}(t) = \langle\ \_\ , 0\rangle$. That is, t for which $\frac{\sin t}{\sqrt{t}} = 0$. Solving gives

$$\frac{\sin t}{\sqrt{t}} = 0 \implies \sin t = 0 \implies t = \pi k : k \in \mathbb{Z}$$

and thus the curve intersects the x-axis at the points

$$\left\{ \left(\frac{(-1)^k}{\sqrt{k\pi}}, 0 \right) : k \in \mathbb{Z}^{\neq 0} \right\}.$$

◆

Question 3.3. Recall that $\text{dom}(\boldsymbol{r}) = \{t : \boldsymbol{r}(t) \text{ is defined}\}$. Thus, when $\boldsymbol{r} =$

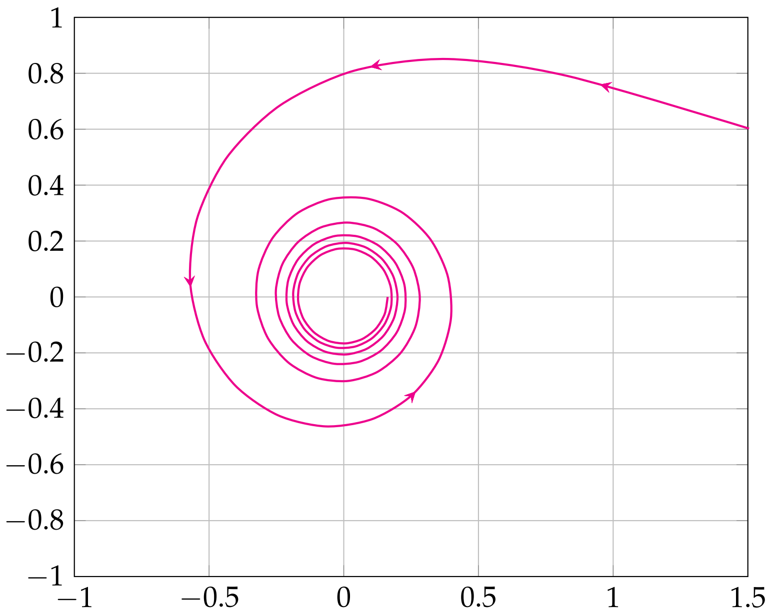


Figure 3.1: $\boldsymbol{r}(t) = \left\langle \frac{\cos t}{\sqrt{t}}, \frac{\sin t}{\sqrt{t}} \right\rangle : t \in (0, 12\pi]$

$\langle f, g \rangle$ we have $\mathrm{dom}(\boldsymbol{r}) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$. What is the domain of $\boldsymbol{r} = \langle t^3, \ln(3-t), \sqrt{t} \rangle$?

ANSWER. We look at the individual domains and find $\mathrm{dom}(\boldsymbol{r}) = \mathrm{dom}(t^3) \cap \mathrm{dom}(\ln(3-t)) \cap \mathrm{dom}(\sqrt{t}) = \mathbb{R} \cap (-\infty, 3) \cap [0, \infty) = [0, 3)$ ◆

3.1.1 Limits on 2D curves

The *limit* of a vector-function $\boldsymbol{r}$ is defined as the component function obtained by taking the limits along the individual components of $\boldsymbol{r}$.

Component Limit 2D. If $\boldsymbol{r}(t) = \langle x(t), y(t) \rangle$ then

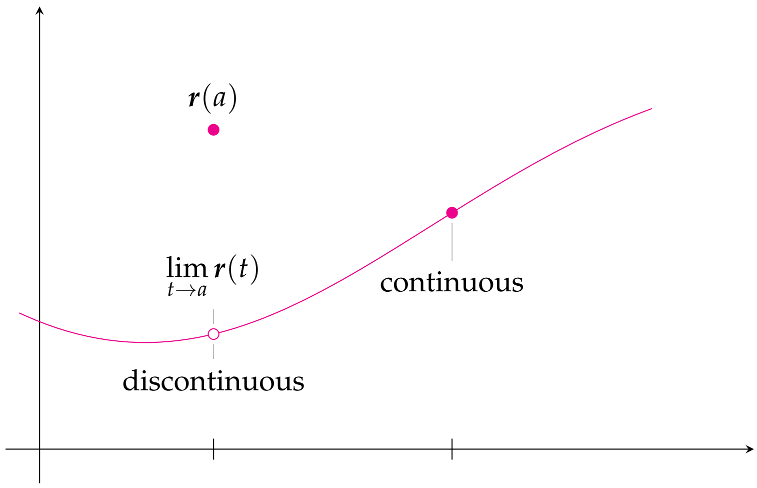
$$\lim_{t \to a} \boldsymbol{r}(t) = \left\langle \lim_{t \to a} x(t), \lim_{t \to a} y(t) \right\rangle$$

provided the individual limits exist.

Question 3.4. Let $\boldsymbol{r} = \left\langle te^{-t}, \frac{\sin t}{t} \right\rangle$ and find $\lim_{t \to 0} \boldsymbol{r}(t)$.

ANSWER. $\lim_{t \to 0} \left\langle te^{-t}, \frac{\sin t}{t} \right\rangle = \left\langle \lim_{t \to 0} te^{-t}, \lim_{t \to 0} \frac{\sin t}{t} \right\rangle = \langle 0, 1 \rangle.$ ◆

Continuous. The vector function $\boldsymbol{r}(t)$ is *continuous* at $a \in \text{dom}(\boldsymbol{r})$ when $\lim_{t\to a} \boldsymbol{r}(t) = \boldsymbol{r}(a)$ and *discontinuous* otherwise.



3.2 *Parametric Curves in* $\mathbb{R}^3$

By adding an additional component for *height* we can define *curves in space*.

Component Function 3D. Let $x(t)$, $y(t)$, and $z(t)$ be *real-valued functions*. Then the *component function* given by these is the *vector-function*

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle : t \in \mathbb{R}.$$

Component Limit 3D. If $\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$ then

$$\lim_{t\to a} \boldsymbol{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \right\rangle$$

provided the individual limits exist.

Let us give the *spiral* some height by adding a third component t. We will get a linear increase in height as the spiral moves inward. See Figure 3.2.

Question 3.5. Where does the spiral limit to as $t \to \infty$?

ANSWER. We deduce $\lim_{t\to\infty} = \left\langle \frac{\cos t}{\sqrt{t}}, \frac{\sin t}{\sqrt{t}}, t \right\rangle = \langle 0, 0, \infty \rangle$ which agrees with our graph. ◆

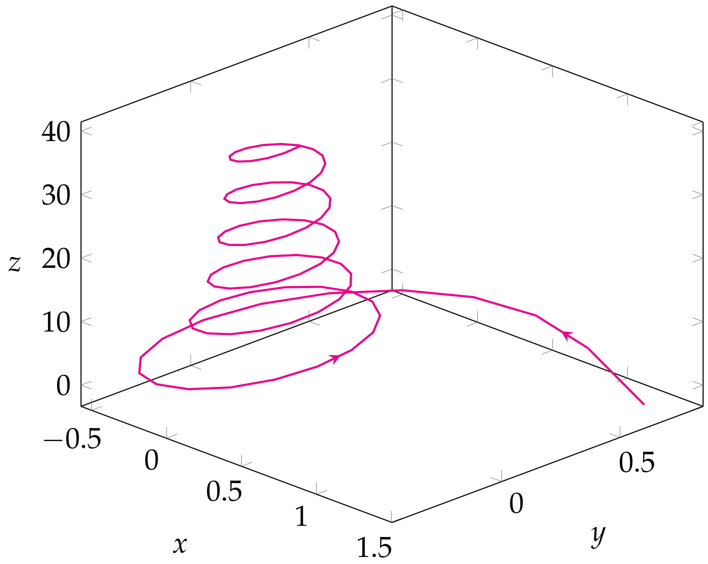


Figure 3.2: $\boldsymbol{r}(t) = \left\langle \frac{\cos t}{\sqrt{t}}, \frac{\sin t}{\sqrt{t}}, t \right\rangle : t \in (0, 12\pi]$.

Question 3.6. Sketch the curve $\boldsymbol{r}(t) = \langle \cos t, \sin t, t \rangle$.

ANSWER. Our strategy here is to see what shape is drawn in the xy-plane (i.e. ignore the z-component and then stretch it.) This shape is the circle and the resulting stretched shape is the *helix*. See Figure 3.3. ◆

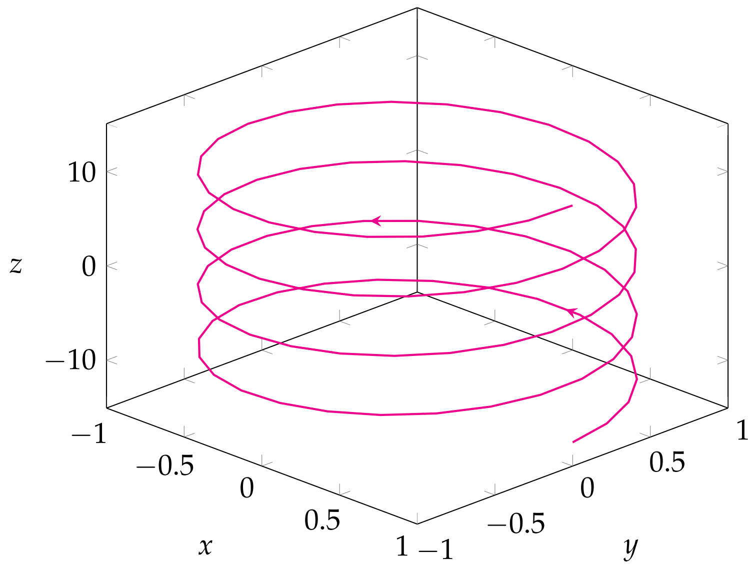


Figure 3.3: $\boldsymbol{r}(t) = \langle \cos t, \sin t, t \rangle : t \in [-2\pi, 2\pi]$

Question 3.7. Find the parametric curve that is left after intersecting the *cylinder* centered at $(1, 0, 0)$ of radius 1 given by $(x-1)^2 + y^2 = 1$ with the *sphere* of radius 2 centered at the origin given by $x^2 + y^2 + z^2 = 2^2$. See Figure 3.4.

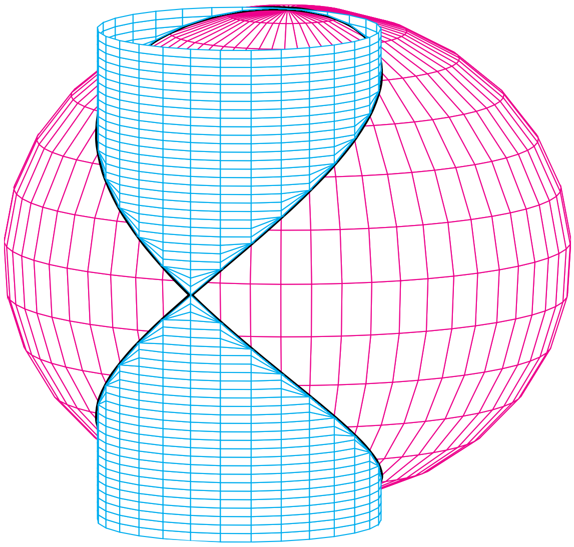


Figure 3.4: Viviani's curve.

ANSWER. We need to determine $\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$ so that we *simultaneously* satisfy

$$(x(t)-1)^2 + y(t)^2 = 1 \qquad \text{and} \qquad x(t)^2 + y(t)^2 + z(t)^2 = 4.$$

We know the cylinder can be parameterized by $(x, y) \leftarrow (1 + \cos t, \sin t)$ so that these equations become

$$\cos^2 t + \sin^2 t = 1 \qquad \text{and} \qquad (1+\cos t)^2 + \sin^2 t + z(t)^2 = 4.$$

Since the first equation is satisfied for any t it only remains to determine what $z(t)$ should be from the second equation. Notice

$$\begin{aligned}
&(1+\cos t)^2 + \sin^2 t + z(t)^2 = 4 \\
&\quad \Longrightarrow \ 1 + 2\cos t + \cos^2 t + \sin^2 t + z(t)^2 = 4 \\
&\quad \Longrightarrow \ z(t)^2 = 4 - 2(1+\cos t) \\
&\quad \Longrightarrow \ z(t)^2 = 4 - 2\left(2\cos^2 \frac{t}{2}\right) && \text{double angle} \\
&\quad \Longrightarrow \ z(t)^2 = 4 - 4\cos^2 \frac{t}{2} \\
&\quad \Longrightarrow \ z(t)^2 = 4\sin^2 \frac{t}{2} \\
&\quad \Longrightarrow \ z(t) = 2\sin \frac{t}{2}
\end{aligned}$$

Thus the parameterization of the intersection is given by

$$\boldsymbol{r}(t) = \left\langle 1 + \cos t, \sin t, 2\sin \frac{t}{2} \right\rangle$$

as illustrated in Figure 3.5. ◆

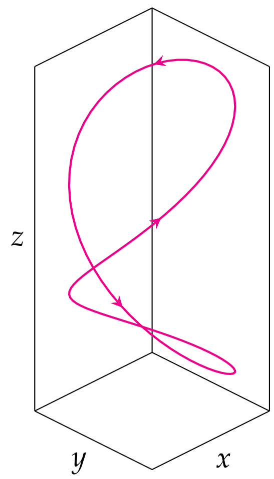
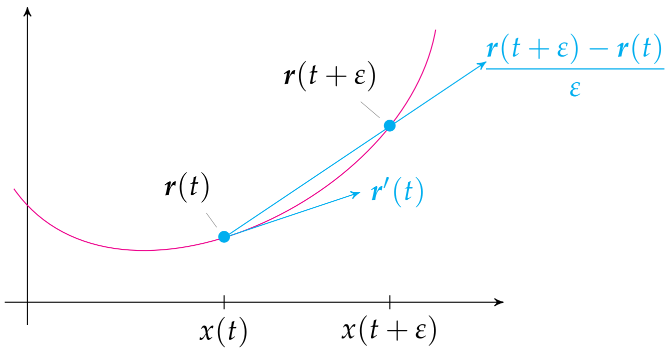


Figure 3.5:
$\boldsymbol{r}(t) = \left\langle 1 + \cos t, \sin t, 2\sin\frac{t}{2} \right\rangle$

3.3 Tangent Lines on Space Curves

From limits we can build a *tangent vector* at t on $\boldsymbol{r}(t) = \langle x(t), y(t) \rangle$:



We do not (yet) care about the magnitude of $\boldsymbol{r}'$, just that it is pointing in the tangent direction.

Question 3.8. What is the *tangent vector* at $(1,1)$ on the parabola $\boldsymbol{r}(t) = \langle t, t^2 \rangle$? (See Figure 3.6.)

ANSWER. The tangent vector is given by the limit

$$\lim_{\varepsilon \to 0} \frac{\boldsymbol{r}(1+\varepsilon) - \boldsymbol{r}(1)}{\varepsilon}$$
$$= \lim_{\varepsilon \to 0} \frac{\langle 1+\varepsilon, (1+\varepsilon)^2 \rangle - \langle 1, 1 \rangle}{\varepsilon} = \lim_{\varepsilon \to 0} \left\langle \frac{\varepsilon}{\varepsilon}, \frac{2\varepsilon + \varepsilon^2}{\varepsilon} \right\rangle$$
$$= \lim_{\varepsilon \to 0} \langle 1, 2 + \varepsilon \rangle = \langle 1, 2 \rangle$$

For one step forward in x we move up two steps in y. ◆

This idea of tangent vector applies to component functions of higher dimension as well.

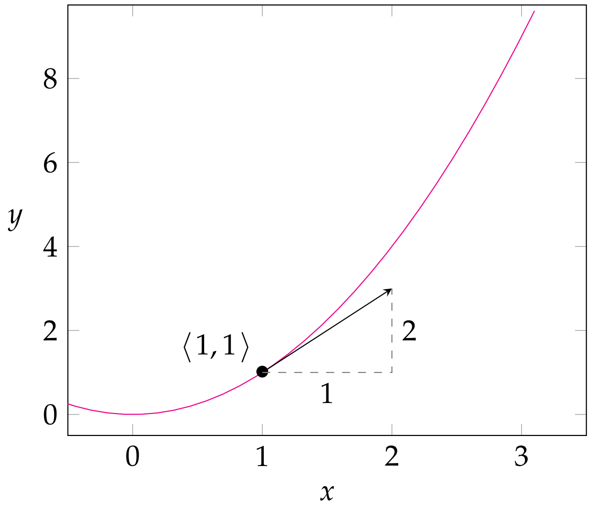


Figure 3.6: The *tangent line* is given by the *parameterization* $\langle 1,1\rangle + t\,\langle 2,1\rangle$.

Tangent Vector. Let $\boldsymbol{r}$ be a component function defining a curve. Then its *tangent vector* at point $\boldsymbol{r}(t)$ is given by

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{r}'(t) := \lim_{\varepsilon\to 0} \frac{\boldsymbol{r}(t+\varepsilon) - \boldsymbol{r}(t)}{\varepsilon}.$$

Proposition 3.9. $\boldsymbol{r} = \langle x, y, z\rangle \implies \dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \left\langle \dfrac{\mathrm{d}x}{\mathrm{d}t}, \dfrac{\mathrm{d}y}{\mathrm{d}t}, \dfrac{\mathrm{d}z}{\mathrm{d}t} \right\rangle.$

PROOF. By definition we have

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &= \lim_{\varepsilon\to 0} \frac{\boldsymbol{r}(t+\varepsilon) - \boldsymbol{r}(t)}{\varepsilon} \\
&= \left\langle \lim_{\varepsilon\to 0} \frac{x(t+\varepsilon) - x(t)}{\varepsilon}, \lim_{\varepsilon\to 0} \frac{y(t+\varepsilon) - y(t)}{\varepsilon}, \lim_{\varepsilon\to 0} \frac{z(t+\varepsilon) - z(t)}{\varepsilon} \right\rangle \\
&= \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle.
\end{aligned}$$

■

Tangent Line. The *parametric equation* of the *tangent line* of $\boldsymbol{r}$ at $a \in \operatorname{dom}(\boldsymbol{r})$ is given by

$$\boldsymbol{r}(a) + t\boldsymbol{r}'(a) : t \in \mathbb{R}.$$

Question 3.10. What is the tangent line on $\boldsymbol{r}(t) = \langle \sin t, 2\cos t, t\rangle$ (the *elliptical helix*) at $\boldsymbol{r}(\pi/2)$?

ANSWER. Note $\boldsymbol{r}' = \langle \cos t, -2\sin t, 1\rangle$ so the tangent line is given by $\boldsymbol{r}(\pi/2) + t\,\boldsymbol{r}'(\pi/2) = \langle 1, 0, \pi/2\rangle + t\langle 0, -2, 1\rangle = \langle 1, -2t, \pi/2 + t\rangle$. ◆

3.4 Parametric Surfaces

A general surface in $\mathbb{R}^3$ can be descried by a vector valued functions of *two parameters*.

Parametric Surface 3D. Let x, y, and z be *component functions* from $\mathbb{R}^2 \to \mathbb{R}$. The *parametric surface* defined by these components is given by

$$\boldsymbol{r}(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle .$$

Elliptical Paraboloid. An *elliptical paraboloid* is a paraboloid with elliptical cross section given by the equation:

$$\frac{z}{c} = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

and can be parameterized by

$$x = a\sqrt{s}\cos t \qquad y = b\sqrt{s}\sin t \qquad z = s.$$

with $t \in [0, 2\pi)$ and $s \in [0, \infty)$. Moreover, when $a = b$ then the surface is called an *circular paraboloid*.

Elliptical Cylinder. An *elliptical cylinder* is a cylinder with an elliptical cross section given by the equation

$$1 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

and can be parameterized by

$$x = a\cos s \qquad y = b\sin s \qquad z = t.$$

with $t \in [0, 2\pi)$ and $s \in [0, \infty)$. When $a = b$ then the surface is called a *circular cylinder*.

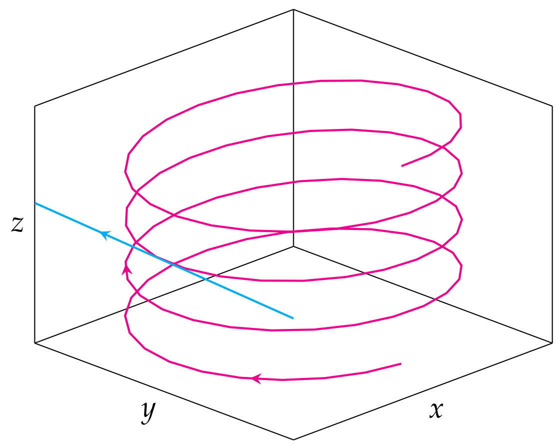


Figure 3.7: The Elliptical Helix $\boldsymbol{r}(t) = \langle \sin t, 2\cos t, t \rangle$

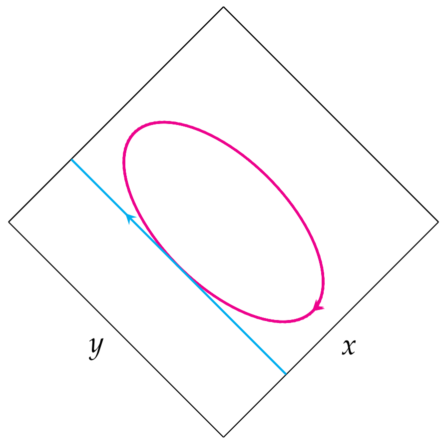


Figure 3.8: Elliptical Helix (View from top)

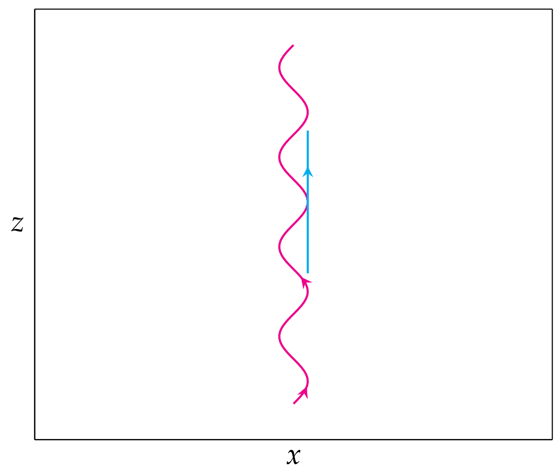


Figure 3.9: Elliptical Helix (View from side)

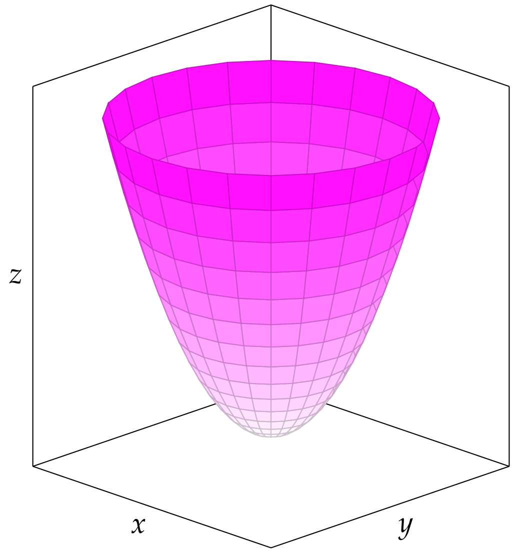


Figure 3.10: Circular Paraboloid

$$z = \left\langle \sqrt{s}\cos t,\ \sqrt{s}\sin t,\ s \right\rangle : (s,t) \in [0,3] \times [0,2\pi).$$

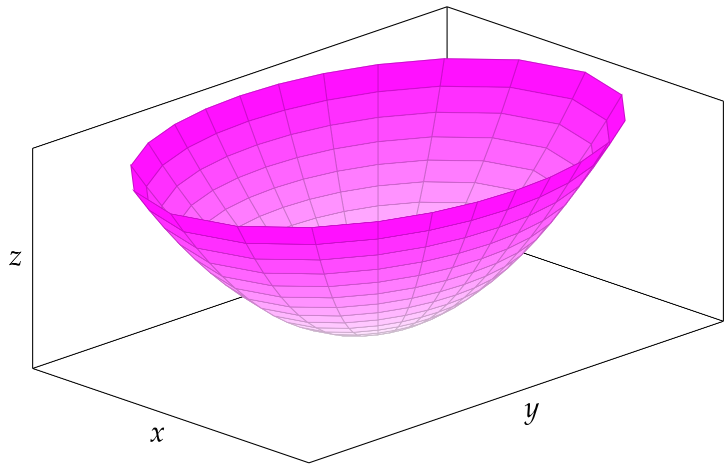


Figure 3.11: Elliptical Paraboloid

$$z = \left\langle 2\sqrt{s}\cos t,\ 3\sqrt{s}\sin t,\ s \right\rangle : (s,t) \in [0,3] \times [0,2\pi).$$

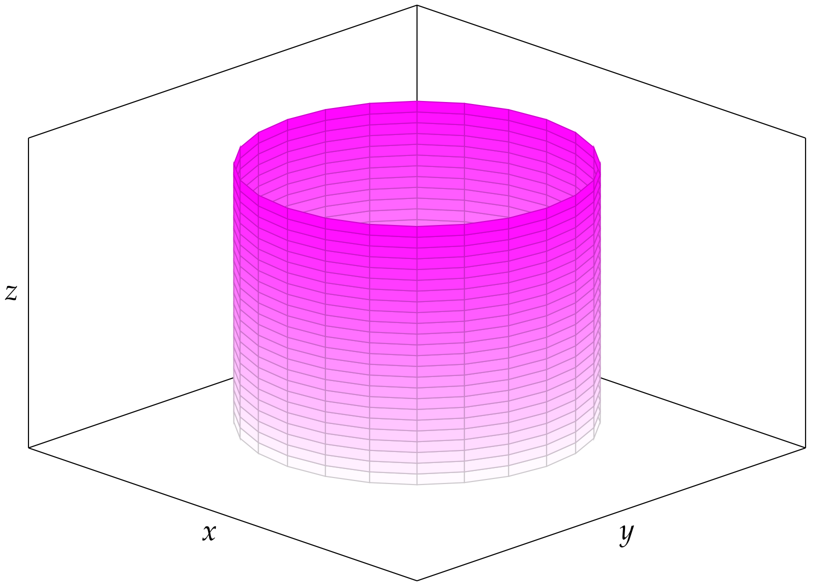


Figure 3.12: Circular Cylinder

$$\langle \cos s,\, \sin s,\, t \rangle : (s,t) \in [0, 2\pi) \times [0,3].$$

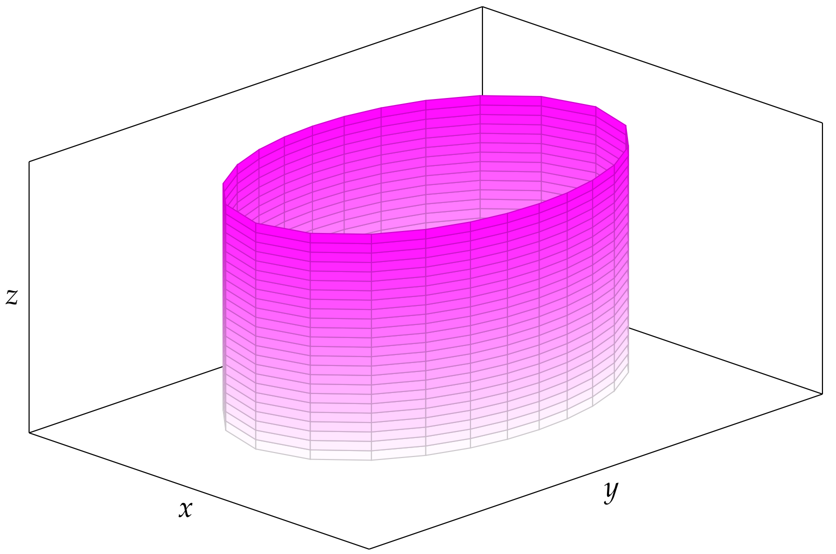


Figure 3.13: Elliptical Cylinder

$$\langle 2\cos s,\, 3\sin s,\, t \rangle : (s,t) \in [0, 2\pi) \times [0,3].$$

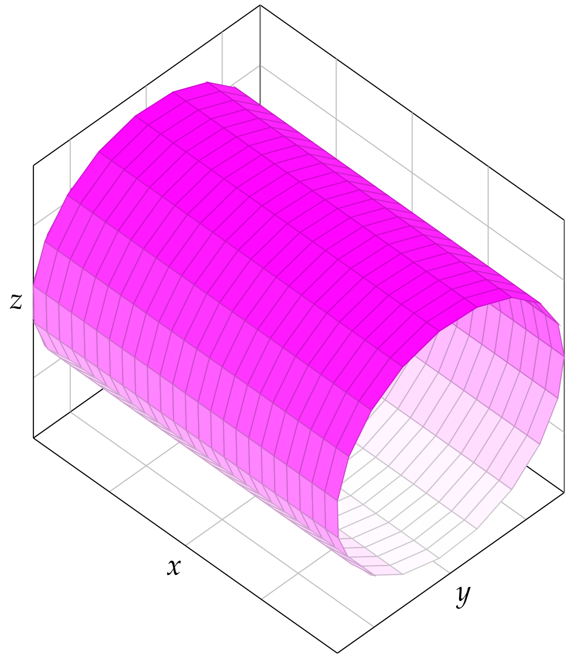


Figure 3.14:

$$\boldsymbol{r}(s,t) = \left\langle s^2,\, t^2,\, s+t \right\rangle : (s,t) \in [-2,2] \times [-2,2].$$

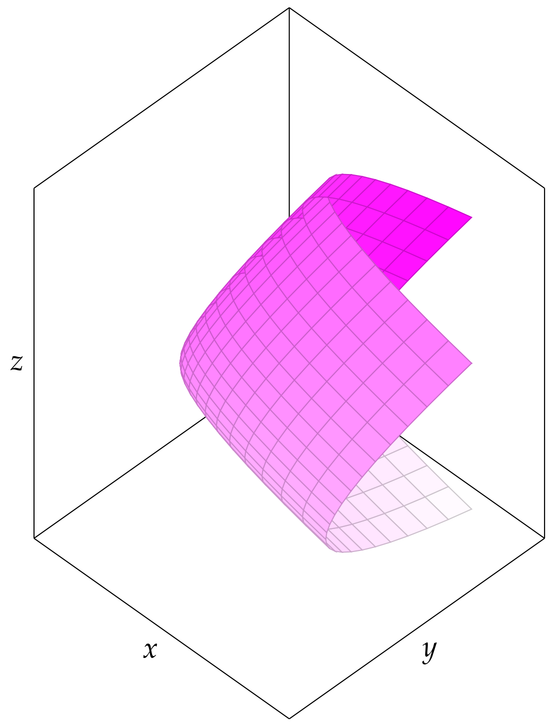


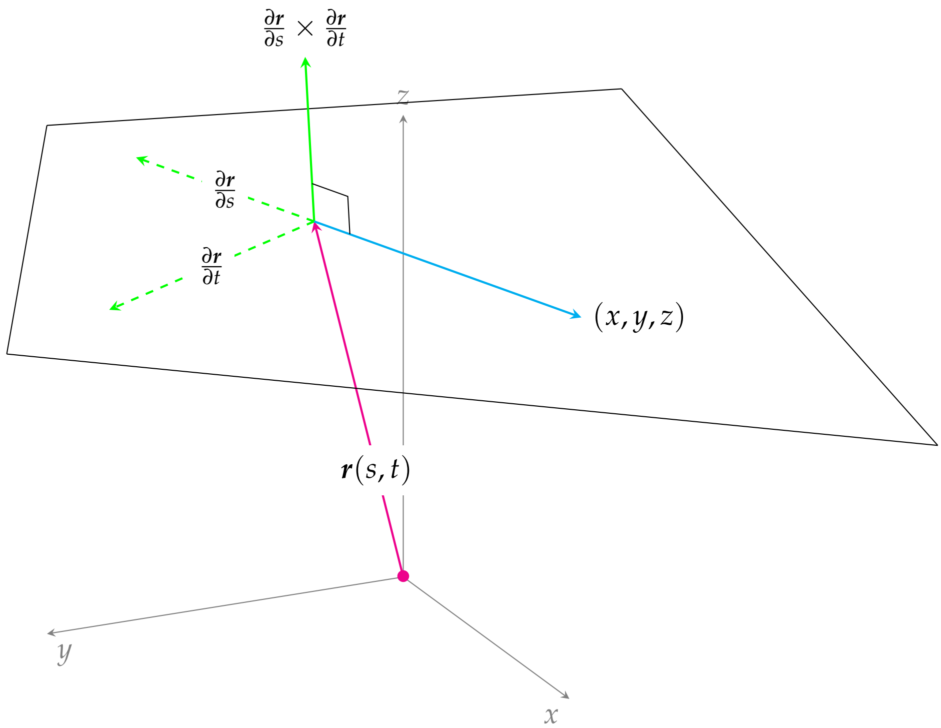
Figure 3.15:

$$\boldsymbol{r}(s,t) = \left\langle s^2,\, t^2,\, s+t \right\rangle : (s,t) \in [-2,2] \times [-2,2].$$

3.5 Tangent Planes on Surfaces

Let us extend the definitions for *tangent vector* and *tangent line* to surfaces.

Taking the partial derivatives with respect to s and t give us *two* tangent vectors that define a *tangent plane*. (We did not have to use partials for space-curves because there was only a single parameter in that setting.) Remember, taking a partial derivative has a *geometric interpretation* of slicing the surface with the plane given by setting either s or t to a constant. We can characterize the tangent plane as all points (x, y, z) which form a perpendicular intersection with the normal to the tangent vectors.



Tangent Plane. Let $\boldsymbol{r}(s, t)$ be a component function defining a surface. Then its *tangent plane* at point $\boldsymbol{r}(a, b) : (a, b) \in \text{dom}(\boldsymbol{r})$ is given by

$$(\langle x, y, z \rangle - \boldsymbol{r}(a, b)) \cdot \frac{\partial \boldsymbol{r}}{\partial s}(a, b) \times \frac{\partial \boldsymbol{r}}{\partial t}(a, b) = 0.$$

Question 3.11. What is the tangent plane at $\boldsymbol{r}(1, 1)$ of the surface $\boldsymbol{r}(u, v) = \langle u^2, v^2, u + v \rangle$? [1]

[1] We switch variables on purpose to reenforce that nothing changes; we must get used to this.

ANSWER. By definition the tangent plane is given by

$$(\langle x, y, z\rangle - \langle 1, 1, 2\rangle) \cdot \frac{\partial \boldsymbol{r}}{\partial u}(1,1) \times \frac{\partial \boldsymbol{r}}{\partial v}(1,1) = 0$$

where $\frac{\partial \boldsymbol{r}}{\partial u} = \langle 2u, 0, 1\rangle$ and $\frac{\partial \boldsymbol{r}}{\partial v} = \langle 0, 2v, 1\rangle$.

We have $\frac{\partial \boldsymbol{r}}{\partial u}(1,1) = \langle 2, 0, 1\rangle$, $\frac{\partial \boldsymbol{r}}{\partial v}(1,1) = \langle 0, 2, 1\rangle$, and

$$\langle 2, 0, 1\rangle \times \langle 0, 2, 1\rangle = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \langle -2, -2, 4\rangle .$$

Thus the tangent plane is given by

$$0 = \langle x-1, y-1, z-2\rangle \cdot \langle -2, -2, 4\rangle = x + y - 2z - 2.$$

◆

4 Double Integrals

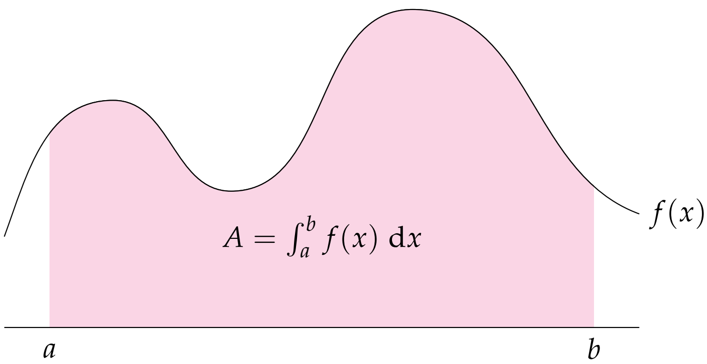
We revisit the *Riemann sums* definition of the integral in $\mathbb{R}^2$ and extend this definition to $\mathbb{R}^3$. This will enable us to find the *volume* bounded by *surfaces* in space.

4.1 Integration in $\mathbb{R}^2$

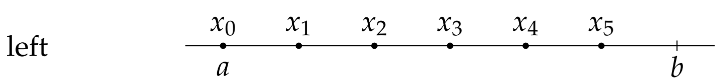
Recall, the *fundamental theorem of calculus* states

$$\frac{\mathrm{d}F}{\mathrm{d}x} = f(x) \implies A = F(b) - F(a)$$

where F is the *anti-derivative* of F satisfying $F' = f$ and



As in Figure 4.2 we approximate the area by summing rectangles with progressively smaller *widths*. As Figure 4.1 illustrates these areas can be made by taking the rectangle's height is taken form the left side of the interval. However, after we partition the interval $[a, b]$ into (say) 6 pieces the x_i (the so-called *sample points*) can be placed *anywhere* in the subinterval:



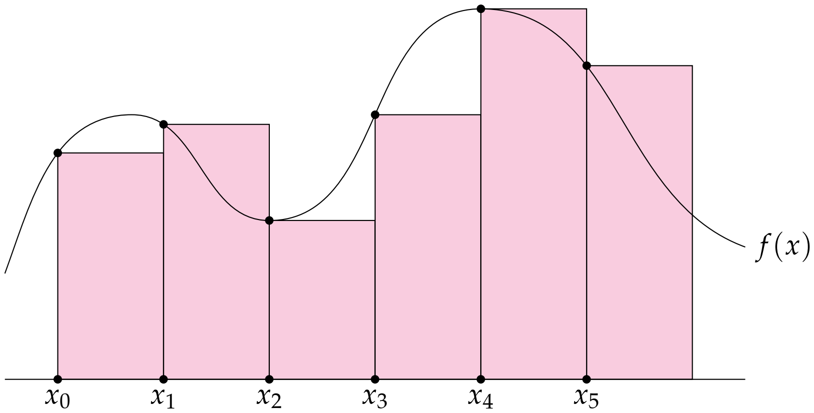


Figure 4.1: Left endpoints.

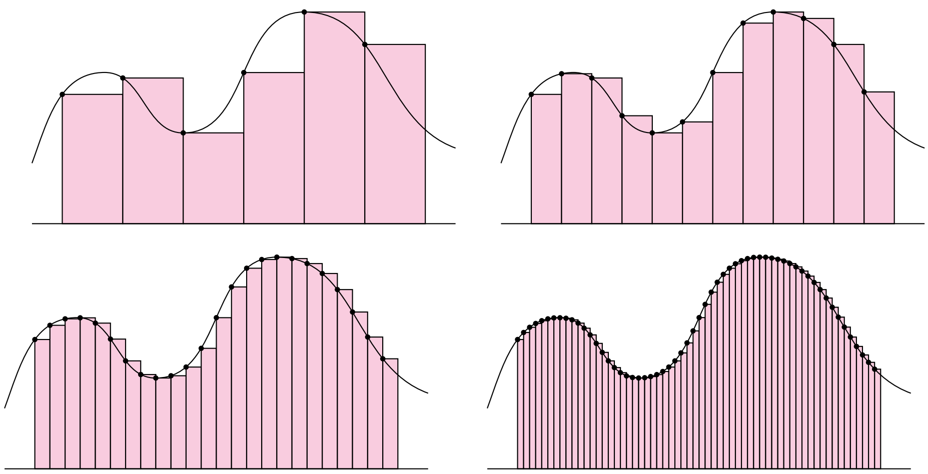
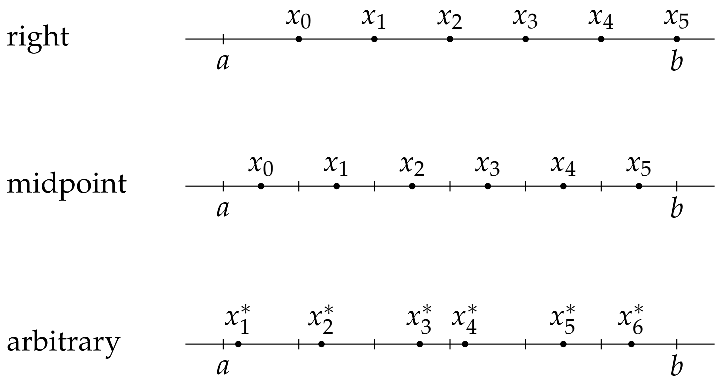
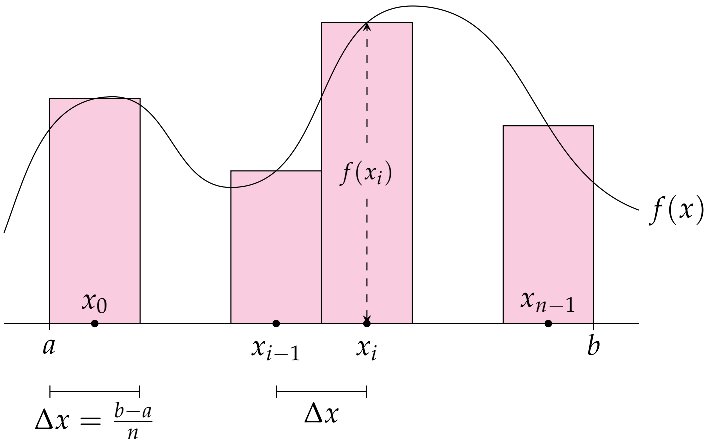


Figure 4.2: Approximating the area under $f(x)$.



Let us switch to *midpoints* and write the Riemann sum for $f(x)$ after dividing the interval $[a, b]$ into n *subintervals*:



We have $x_{i+1} = x_i + \Delta x$ where $x_0 = a + \frac{\Delta x}{2}$ so

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x.$$

Integral. Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function with $[a, b] \subseteq \text{dom}(f)$. Assume further that $[a, b]$ is divided into n-many equally spaced subintervals with x_i^* in the ith interval. Then

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x. \tag{4.1}$$

with $\Delta x = \frac{b-a}{n}$

4.2 Double Integration

Let us extend this *planar* definition of the integral (i.e. summing up the *area* of *rectangles*) to *space* (i.e. summing up the *volumes* of *prisms*). We can form the integral by adding the *volumes* of *rectangular prisms* with progressively small bases. The volume of this rectangular prism is $f(x^*, y^*)\Delta x \Delta y$ as illustrated in Figure 4.3.

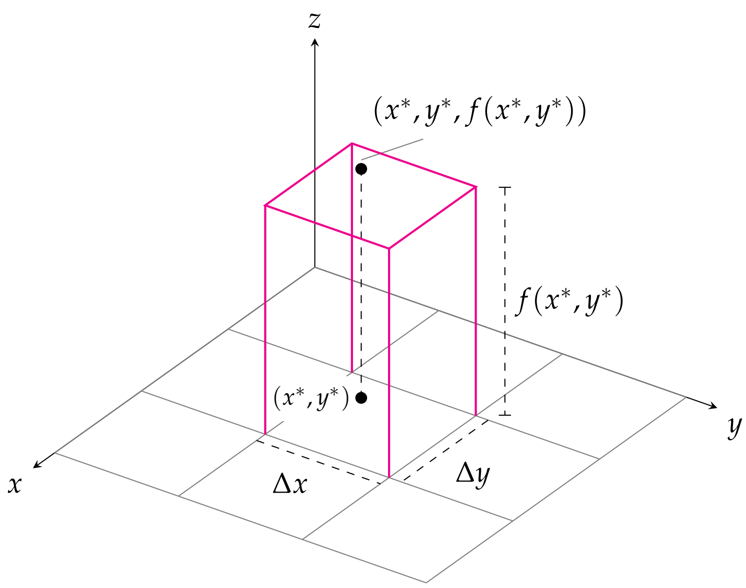


Figure 4.3: The volume of this prism is $\Delta x\, \Delta y\, f(x^*,\, y^*)$.

Rectangular Interval. Let $a < b$ and $c < d$ such that $a, b, c, d \in \mathbb{R}$. Then the *rectangular interval* $[a, b] \times [c, d]$ is given by

$$R := [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b,\, c \leq y \leq d\}\,.$$

Question 4.1. On the *line* we can use the *left*, *right*, or *midpoint* of an interval for the height of the rectangle. What are similar options for ‘intervals’ on the *plane*?

ANSWER. It depends what we mean by *interval on the plane*. For *sub-rectangular* intervals $[a, b] \times [c, d]$ we have *five* options: four corners and the midpoint of the rectangle — see Figure 4.6. ◆

Again, because it does not matter what point we choose — as long as the point is *somewhere* in the rectangle — we simply say this *sample point* is (x^*, y^*) as illustrated in Figure 4.3.

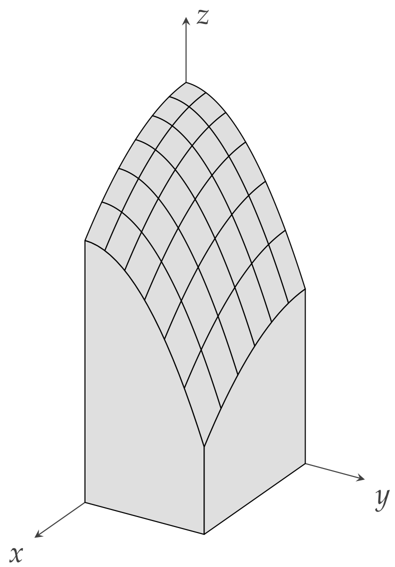


Figure 4.4: The *volume* bounded by the surface

$$f(x,y) = 16 - x^2 - 2y^2$$

in the *square* $[0,2] \times [0,2]$.

Double Integral. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real function with $[a,b] \times [c,d] \subseteq \text{dom}(f)$. Assume further that $R = [a,b] \times [c,d]$ is divided into nm-many equally sized sub-rectangles with (x_i^*, y_i^*) in the (i, j)th sub-rectangle. Then

$$\iint_R f(x)\,\mathrm{d}A = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta x \Delta y. \tag{4.2}$$

Question 4.2. *Estimate* the *volume* lying below the surface $z = xy$ and above the rectangle $[0,6] \times [0,4]$ by dividing the x and y intervals into 3 and 2 subintervals respectively. Use midpoints for your approximation.

ANSWER. Since $\Delta x = \frac{6-0}{3} = 2$ and $\Delta y = \frac{4}{2} = 2$, the *sum of the rectangular prisms* is

$$\sum_{(x,y)\in \boldsymbol{x}\times\boldsymbol{y}} f(x,y)\,\Delta x\,\Delta y = \sum_{(x,y)\in \boldsymbol{x}\times\boldsymbol{y}} 4xy.$$

It suffices then to determine $\boldsymbol{x} \times \boldsymbol{y}$: the collection of midpoints. These midpoints are given by $\boldsymbol{x} = \{1, 3, 5\}$ and $\boldsymbol{y} = \{1,3\}$ as in Figure 4.7. Thereby $\boldsymbol{x} \times \boldsymbol{y} = \{(1,1),(1,3),(3,1),(3,3),(5,1),(5,3)\}$ and so the Riemann sum expands as

$$\sum_{(x,y)\in \boldsymbol{x}\times\boldsymbol{y}} 6xy$$

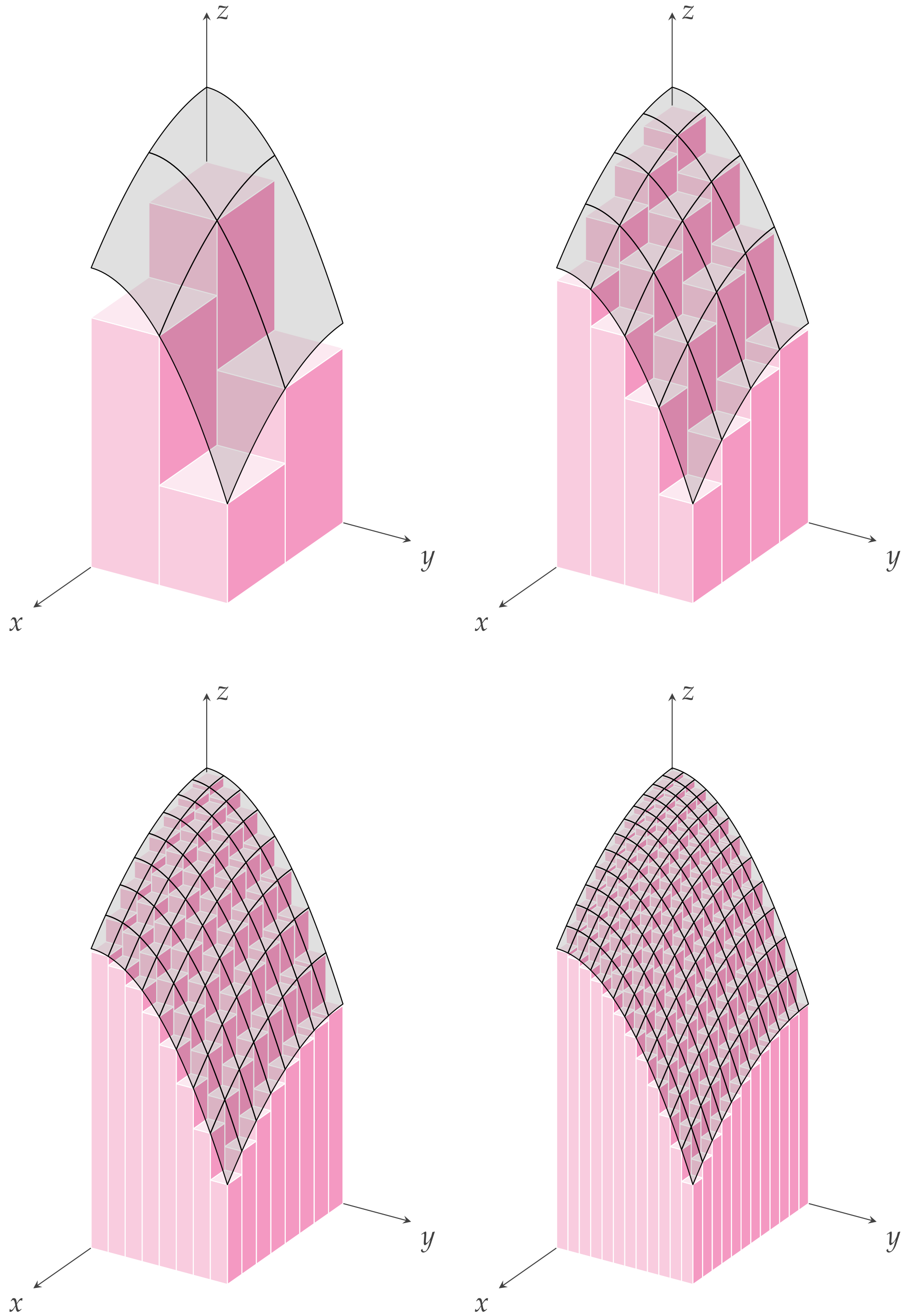


Figure 4.5: A 3D Riemann sum — Successive approximations of the volume bounded by $f(x,y) = 16 - x^2 - 2y^2$ over the square $[0,2] \times [0,2]$.

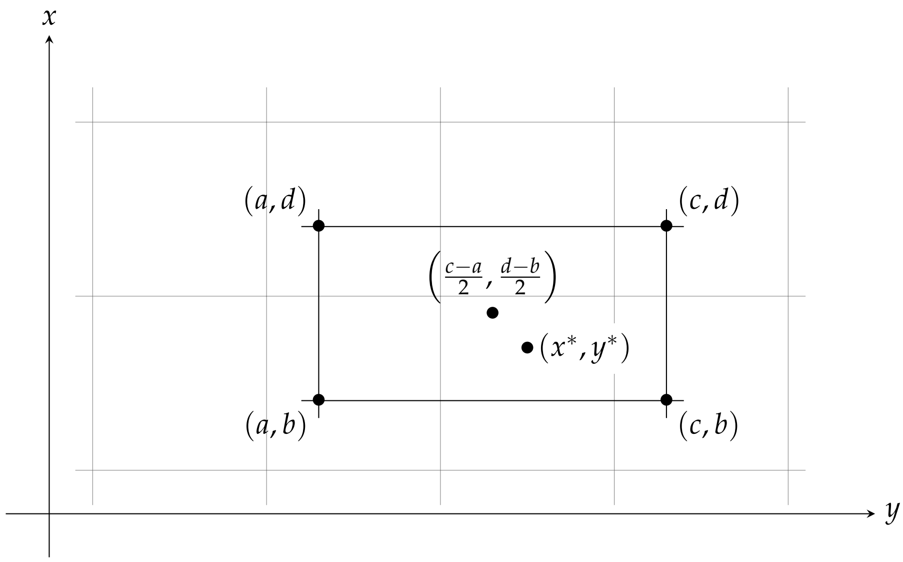


Figure 4.6: A midpoint and sample point from a rectangle.

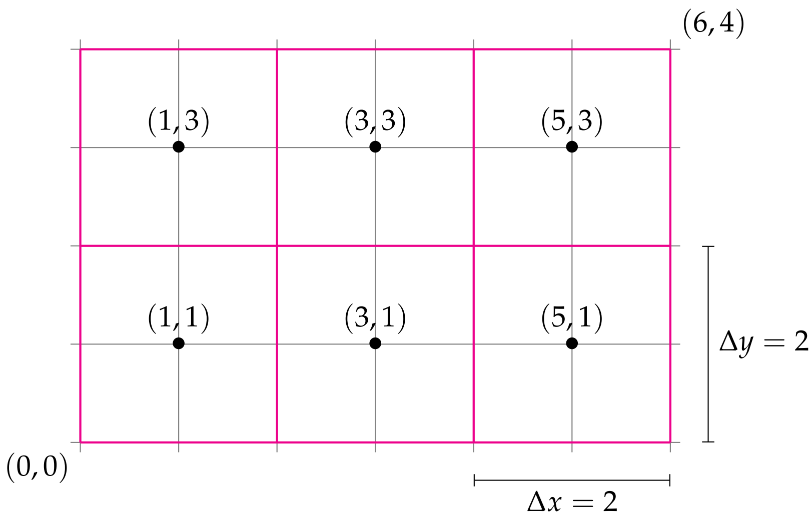


Figure 4.7: The sample midpoints for Question 4.2.

$$= 4\,[(1)(1) + (1)(3) + (3)(1) + (3)(3) + (5)(1) + (5)(3)]$$
$$= 4 \cdot 36 = 144.$$

◆

4.3 Iterated Integrals

It would be impractical to use Riemann sums for computing integrals. We instead use *integrated integral* to *algebraically* determine a double integral by doing two *single-variable* integrals using the myriad of tools developed for doing so.

Iterated Integral. Let $R = [a,b] \times [c,d]$ give a rectangular region in $\operatorname{dom}(f)$ for $f : \mathbb{R} \to \mathbb{R}^2$ a real surface. Then

$$\iint_R f(x,y)\ \mathrm{d}A = \int_a^b \int_c^d f(x,y)\ \mathrm{d}y\ \mathrm{d}x.$$

Theorem 4.3 (Fubini's). Let $R = [a,b] \times [c,d]$ give a rectangle in $\operatorname{dom}(f)$ for $f : \mathbb{R} \to \mathbb{R}^2$ a real surface. If f is continuous on R then

$$\int_a^b \int_c^d f(x,y)\ \mathrm{d}y\ \mathrm{d}x = \int_c^d \int_a^b f(x,y)\ \mathrm{d}x\ \mathrm{d}y.$$

That is, the order of the integrals may be permuted.

Question 4.4. Let $R = [0,2] \times [1,2]$. Calculate $\iint_R (x - 3y^2)\ \mathrm{d}A$.

ANSWER. We have

$$\iint_R (x - 3y^2)\ \mathrm{d}A = \int_0^2 \int_1^2 (x - 3y^2)\ \mathrm{d}y\ \mathrm{d}x.$$

The inner integral evaluates

$$\int_1^2 (x - 3y^2)\ \mathrm{d}y = \left[xy - y^3\right]_{y=1}^{y=2} = (2x - 2^3) - (x - 1^3) = x - 9$$

which implies

$$\int_0^2 \int_1^2 (x - 3y^2)\ \mathrm{d}y\ \mathrm{d}x$$
$$= \int_0^2 x - 9\ \mathrm{d}x = \left[\frac{x^2}{2} - 9x\right]_{x=0}^{x=2} = \left(\frac{2^2}{2} - 18\right) = -16$$

◆

Exercise 4.1. Verify Fubini's theorem by repeating Question 4.4 with the $\mathrm{d}x$ and $\mathrm{d}y$ exchanged.

Question 4.5. Calculate $\int_0^3 \int_0^2 y\sqrt{x+y^2}\,\mathrm{d}y\,\mathrm{d}x$.

ANSWER. Using substitution: $u = x + y^2 \implies \mathrm{d}u = 2y\mathrm{d}y \implies \mathrm{d}y = \frac{\mathrm{d}y}{2y}$.

$$\int_{y=0}^{y=2} y\sqrt{x+y^2}\,\mathrm{d}y = \int_{u=x}^{u=x+2^2} y\sqrt{u}\,\frac{\mathrm{d}u}{2y} = \frac{1}{2}\int_{u=x}^{u=x+2^2} \sqrt{u}\,\mathrm{d}u$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=x}^{u=x+2^2} = \frac{1}{3}\left[(x+y^2)^{\frac{3}{2}}\right]_{y=0}^{y=2} = \frac{1}{3}\left((x+4)^{\frac{3}{2}} - (x-0)^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}\left((x+4)^{\frac{3}{2}} - x^{\frac{3}{2}}\right).$$

All that remains is to find $\int_0^3 \frac{1}{3}\left((x+4)^{\frac{3}{2}} - x^{\frac{6}{3}}\right)\mathrm{d}x$:

$$\frac{1}{3}\int_0^3 \left((x+4)^{\frac{3}{2}} - x^{\frac{3}{2}}\right)\mathrm{d}x$$

$$= \frac{1}{3}\left[\frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^3 = \frac{1}{3}\cdot\frac{2}{5}\left[\left(7^{\frac{5}{2}} - 3^{\frac{5}{2}}\right) - (0-0)\right]$$

$$= \frac{2}{15}\left(7^{\frac{5}{2}} - 3^{\frac{5}{2}}\right).$$

◆

Question 4.6. Calculate $\int_1^2 \int_0^\pi y\sin(xy)\,\mathrm{d}y\,\mathrm{d}x$.

ANSWER. Notice that $\int_0^\pi y\sin(xy)\,\mathrm{d}y$ is hard (would require integration by parts) *but* $\int_1^2 y\sin(xy)\,\mathrm{d}x$ is relatively easy:

$$\int_1^2 y\sin(xy)\,\mathrm{d}x = [-\cos(xy)]_{x=1}^{x=2} = \cos(y) - \cos(2y).$$

Moreover

$$\int_0^\pi \cos(y) - \cos(2y)\,\mathrm{d}y$$

$$= \left[\sin(y) - \frac{\sin(2y)}{2}\right]_{y=0}^{y=\pi} = \sin(\pi) - \frac{1}{2}\sin(2\pi) = 0.$$

and thus the integral is 0. ◆

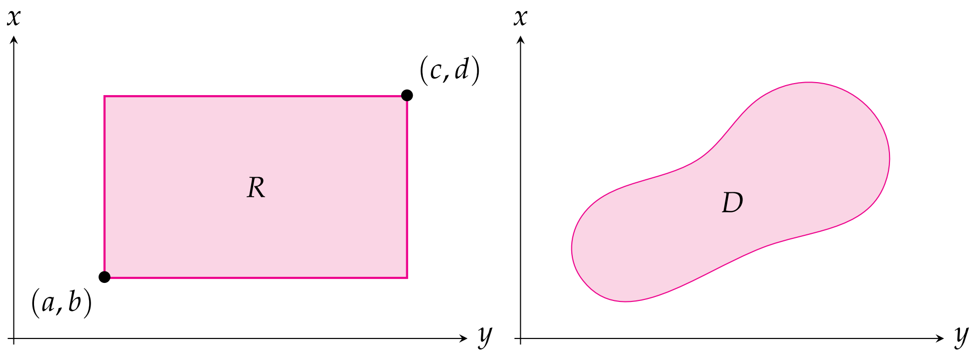
Moral. Sometimes a double integral is easier to calculate when the iterated integrals are swapped.

4.4 Double Integrals Over General Regions

We can integrate over *rectangular regions* R because

$$\iint_R f(x,y)\, \mathrm{d}x\, \mathrm{d}y = \int_a^b \int_c^d f(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

But can we integrate over *general regions* D like the ones below?



It depends on D.

Type I Region. A *Type I* region is a region $D \subseteq \mathbb{R}^2$ given by

$$D = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

for g_1, $g_2 \in \mathbb{R} \to \mathbb{R}$ real valued functions. See Figure 4.8.

Proposition 4.7. If f is continuous on the Type I region D then

$$\iint\limits_D = f(x,y)\, \mathrm{d}A = \int\limits_a^b \int\limits_{g_1(x)}^{g_2(x)} f(x,y)\, \mathrm{d}y\, \mathrm{d}x.$$

Question 4.8. Let D be the region bounded by $2x^2$ and $y = 1 + x^2$ and evaluate $\iint_D (x + 2y)\, \mathrm{d}A$.

ANSWER. D is given by Figure 4.9 or explicitly

$$D = \left\{(x,y) : x \in [-1,1],\ y \in \left[2x^2, 1 + x^2\right]\right\}.$$

(Remember the interval $[a,b]$ must have $a \leq b$ to be nonempty.)

$$\iint_D (x + 2y)\, \mathrm{d}A$$
$$= \int_{x=-1}^{x=1} \int_{y=2x^2}^{y=1+x^2} (x + 2y)\, \mathrm{d}y\, \mathrm{d}x = \int_{x=-1}^{x=1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2}\, \mathrm{d}x$$

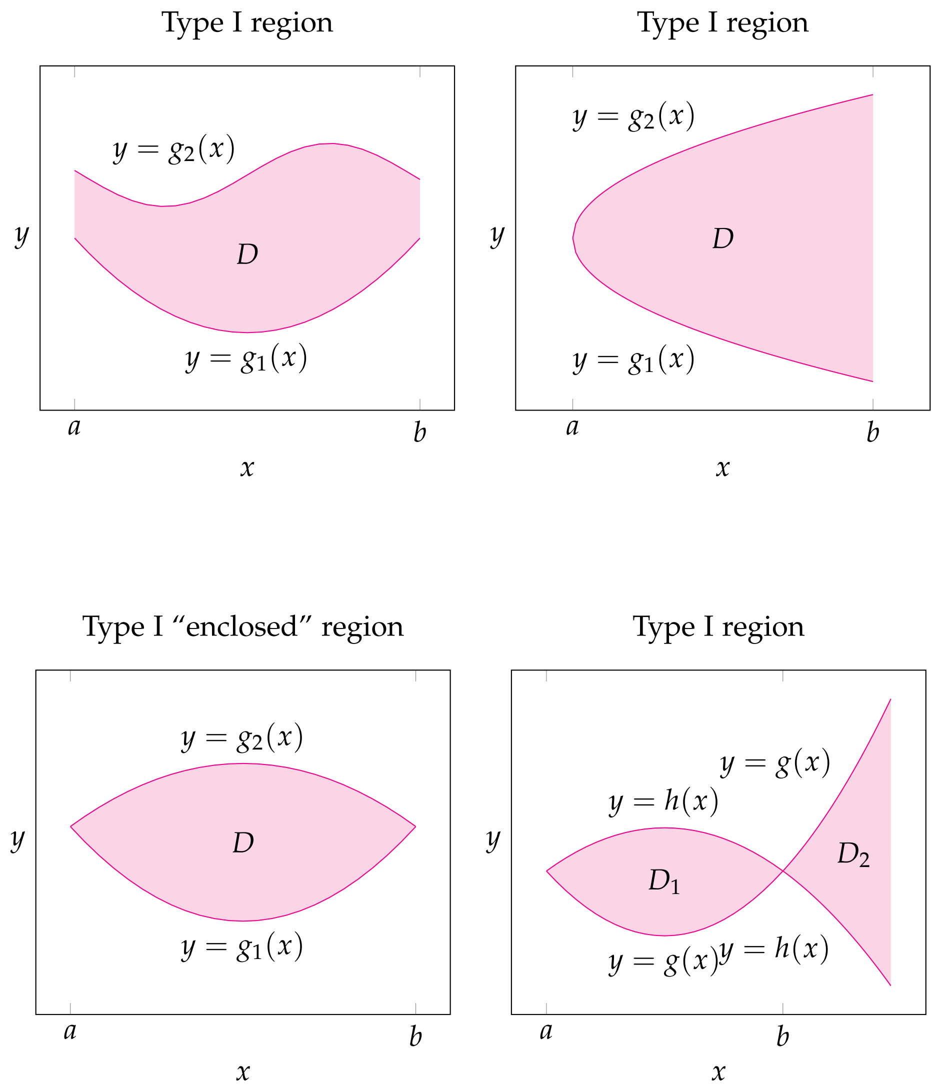


Figure 4.8: Various regions in $\mathbb{R}^2$.

$$= \int_{x=-1}^{x=1} \left[x(1+x^2) + (1+x^2)^2\right] - \left[x(2x^2) + (2x^2)^2\right] \, \mathrm{d}x$$
$$= \int_{x=-1}^{x=1} -3x^4 - x^3 + 2x^2 + x + 1 \, \mathrm{d}x$$
$$= \left[-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x\right]_{x=-1}^{x=1} = \frac{32}{15}.$$

◆

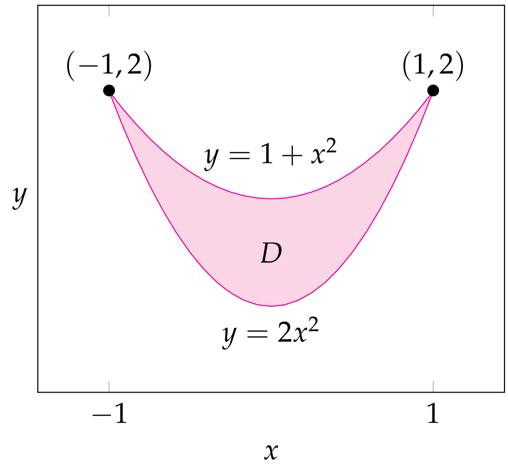


Figure 4.9: For Question 4.9.

Question 4.9. Let D be the region *enclosed by* $y = 0$ and $y = \sqrt{4 - x^2}$ and evaluate $\iint_D x^2 y \, \mathrm{d}A$.

Note D could have been equivalently defined as "the region enclosed by the y-axis and the top half of the circle of radius 2."

ANSWER. We first determine our region — See Figure 4.10. And thus $D = \left\{(x, y) : x \in [-2, 2], y \in [0, \sqrt{4 - x^2}]\right\}$.

$$\iint_D (xy^2) \, \mathrm{d}A$$
$$= \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} xy^2 \, \mathrm{d}y \, \mathrm{d}x = \int_{-2}^{2} \left[\frac{xy^3}{3}\right]_{y=0}^{y=\sqrt{4-x^2}} \mathrm{d}x$$
$$= \frac{1}{3} \int_{-2}^{2} \left(x \left(\sqrt{4 - x^2}\right)^3 - 0\right) \, \mathrm{d}x = \frac{1}{3} \int_{-2}^{2} x \left(4 - x^2\right)^{\frac{3}{2}} \, \mathrm{d}x$$
$$= \frac{1}{3} \left[-\frac{1}{5}(4 - x^2)^{\frac{5}{2}}\right]_{-2}^{2} = 0.$$

◆

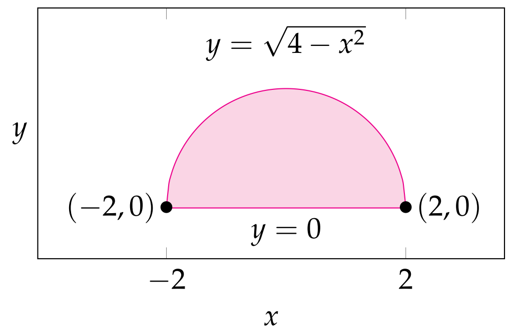


Figure 4.10: For Question 4.4

Type II Region. A *Type II* region is a region $D \subseteq \mathbb{R}^2$ given by

$$D = \{(x, y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

for $g_1, g_2 \in \mathbb{R} \to \mathbb{R}$ real valued functions. See Figure 4.11.

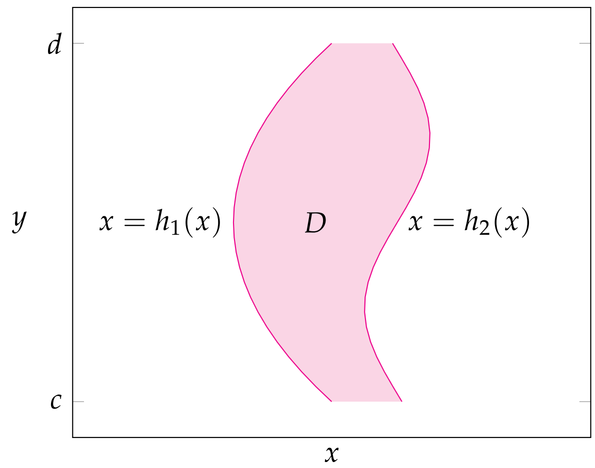
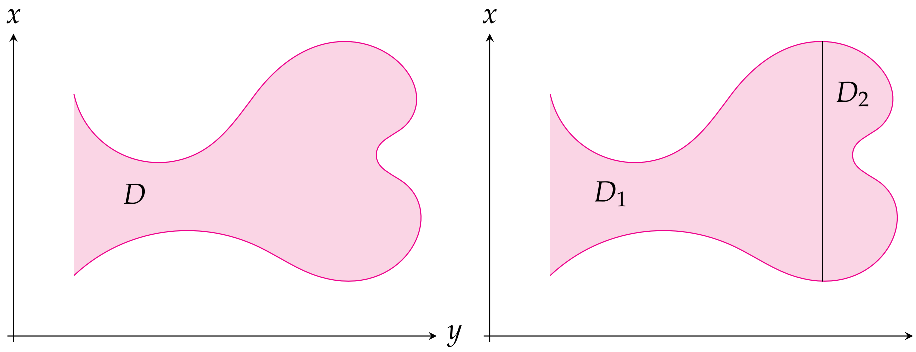


Figure 4.11: A Type II Region

Proposition 4.10. If f is continuous on the Type II region D then

$$\iint_D = f(x, y)\, \mathrm{d}A = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

Regions like D (below) can be divided into two regions D_1 and D_2 so that D_1 is Type I and D_2 is Type II.



Proposition 4.11. Let $D = D_1 \cup D_2$ such that D_1 and D_2 do not overlap except (perhaps) on their boundary. Then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$

The upshot is that we can take integrals over regions that can be *partitioned* into Type I and II regions.

Question 4.12. Let D be the region enclosed by $y = x - 1$ and $y^2 = 2x + 6$. What are the Type I and Type II descriptions of D?

ANSWER. Again our first step is to visualize the enclosed region. See Figure 4.14.

TYPE I

$$D_1 = \left\{ (x,y) : x \in [-3,-1],\ y \in [-\sqrt{2x+6},\ \sqrt{2x+6}] \right\},$$
$$D_2 = \left\{ (x,y) : x \in [-1,5],\ y \in [x-1,\ \sqrt{2x+6}] \right\}.$$

TYPE II $D = \{(x,y) : y \in [-1,4],\ x \in [\tfrac{1}{2}y^2 - 6,\ y+1]\}$.

We have two options for integrating a function $f(x,y)$ over this region: Using the *Type I* region

$$\iint_D f(x,y)\,\mathrm{d}A = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

or using the (better) *Type II* region

$$\iint_D f(x,y)\,\mathrm{d}A = \int_{-1}^{4} \int_{\frac{1}{2}y^2-6}^{y+1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

◆

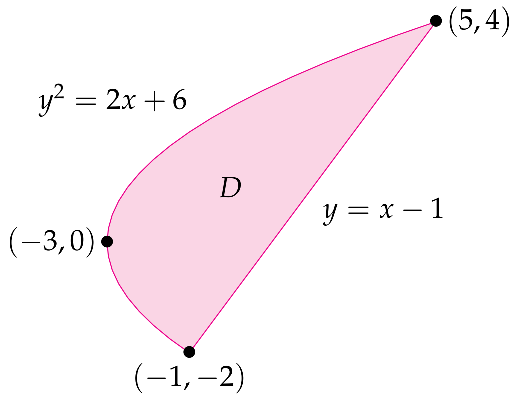


Figure 4.12: For Question 4.12.

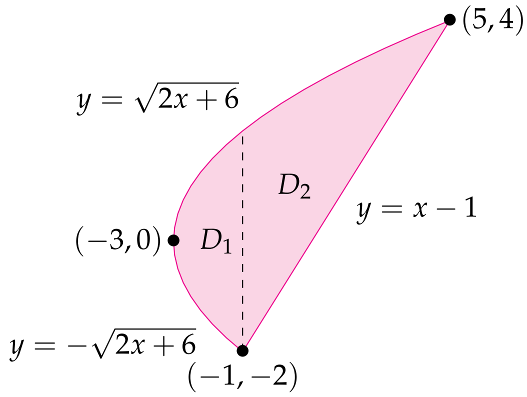


Figure 4.13: For Question 4.12 Type I.

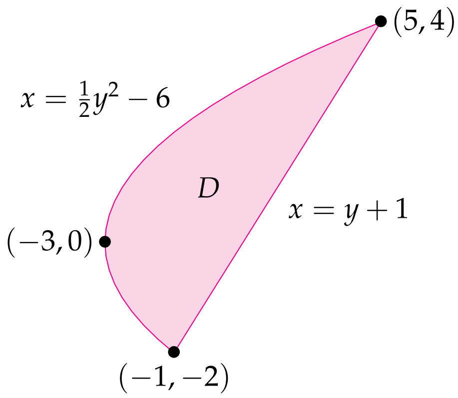


Figure 4.14: For Question 4.12 Type II.

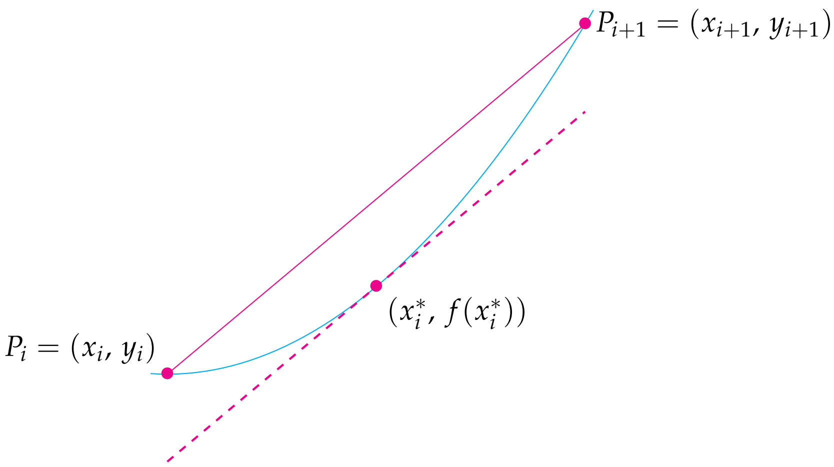
4.5 Arc Length

Consider fitting a piece of string along a curve as in Figure 4.15 and then measuring that string against a ruler — this is our notion of *length of a curve*. We calculate this length by segmenting the curve into successively shorter arcs and then summing the lengths of these arcs. This is similar to what we did with area to compute the area under the curve.

The *length* L of the curve C is *approximately* the sum of the length of the line segments P_iP_{i+1} over $[a,b]$. For N segments this is given by

$$L(N) = \sum_{i=1}^{N} |P_{i-1}P_i|.$$

where $|P_{i-1}P_i|^2 = (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2$:



Mean value theorem assures there is $x_i^* \in [x_{i-1}, x_i]$ such that

$$\begin{aligned} f'(x_i^*) &= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \\ &\implies f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}) \\ &\implies \Delta y_i = f'(x_i^*)\Delta x. \end{aligned}$$

Note it is Δx and not Δx_i because these x_i are equally spaced; thus $\Delta x_i = \Delta x_j$ for all i, j. From this we derive

$$\begin{aligned} L(N) &= \sum_{i=1}^{N} |P_{i-1}P_i| = \sum_{i=1}^{N} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sum_{i=1}^{N} \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sum_{i=1}^{N} \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} \\ &= \sum_{i=1}^{N} \sqrt{1 + f'(x_i^*)^2}\,\Delta x. \end{aligned}$$

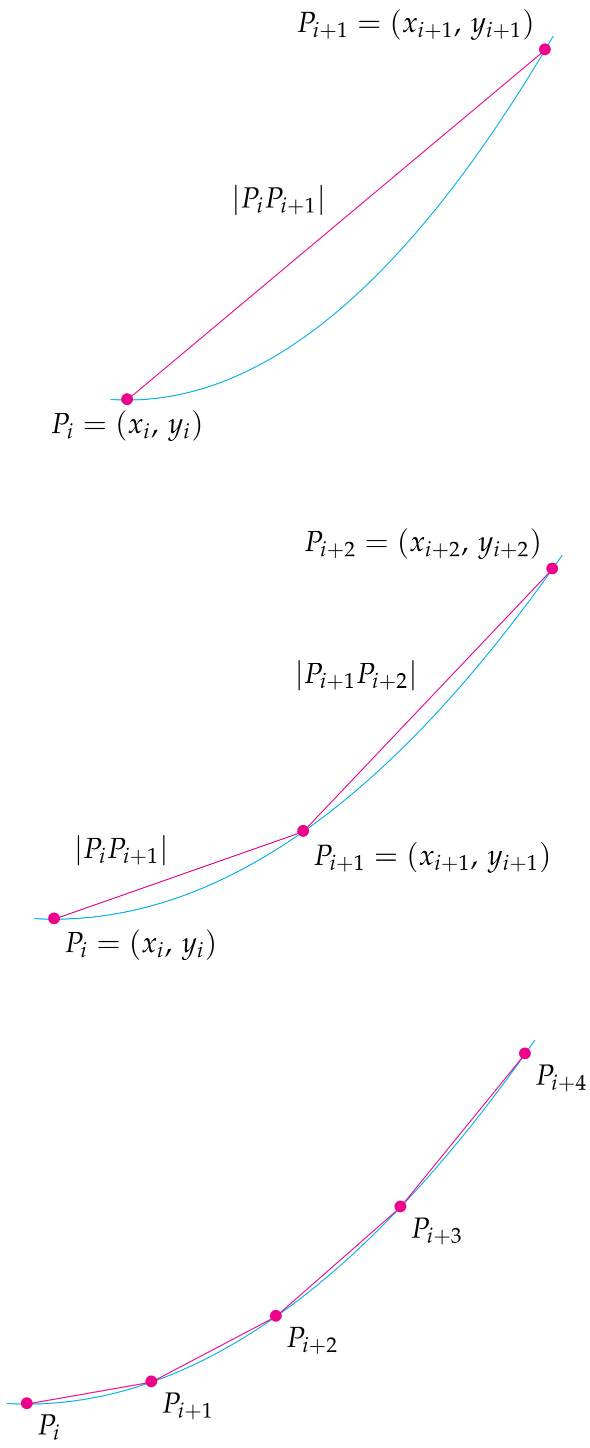


Figure 4.15: Successively better approximations of an arc-length.

and thereby $L(\infty) = \lim_{N\to\infty} \sum_{i=1}^{N} \sqrt{1 + f'(x_i^*)^2}\,\Delta x = \int_a^b \sqrt{1+(f'(x))^2}\,\mathrm{d}x.$

> **Arc Length.** If f' is continuous on $[a, b]$ then the length of the curve $y = f(x)$ over $[a, b]$ is
>
> $$L := \int_a^b \sqrt{1 + (f')^2}\,\mathrm{d}x.$$

Question 4.13. Find the length of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$. See Figure 4.16.

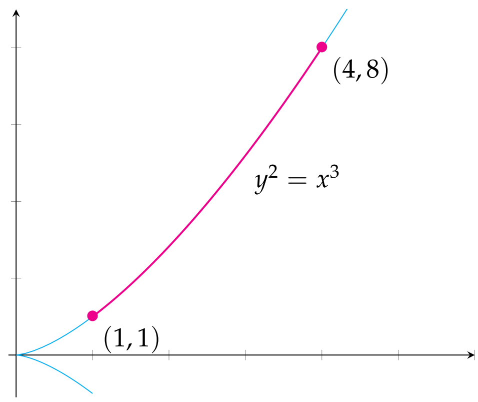


Figure 4.16: $y^2 = x^3$

ANSWER. For the top half of the curve we have $y = x^{\frac{3}{2}}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}x^{\frac{1}{2}}$ which implies the arc length formula gives

$$L = \int_1^4 \sqrt{1 + (f')^2}\,\mathrm{d}x = \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \cdots = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13}).$$

◆

4.6 Surface Area

The surface area of a solid object is a measure of the total area that the surface of the object occupies. We calculate this length by segmenting the surface into successively smaller rectangles as in Figure 4.17 and summing the areas.

If we partition the surface S into nm many sub-rectangles with $P_{ij} = (x_i^*, y_j^*)$ in the ijth partition then the *surface area* of S is

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

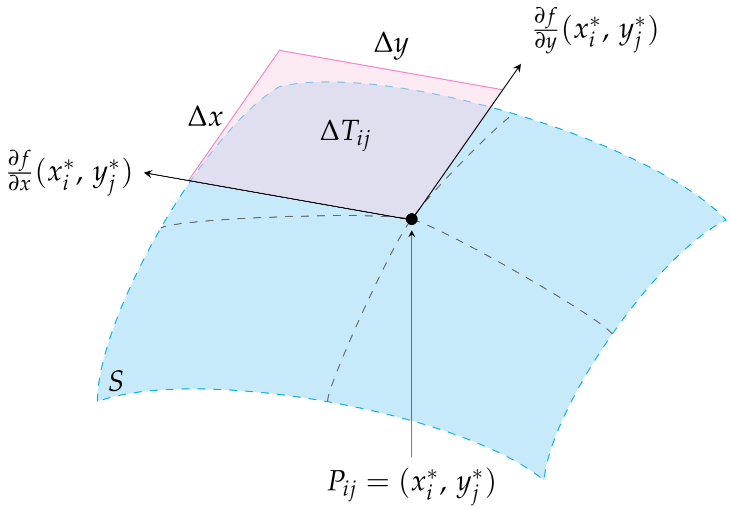


Figure 4.17: $\Delta T_{ij} = \left|\frac{\partial f}{\partial x}(P_{ij})\Delta x \times \frac{\partial f}{\partial y}(P_{ij})\Delta y\right|$ is the area of the pink square.

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \left|\frac{\partial f}{\partial x}(P_{ij})\Delta x \times \frac{\partial f}{\partial y}(P_{ij})\Delta y\right|$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \sqrt{\left(\frac{\partial f}{\partial x}(P_{ij})\right)^2 + \left(\frac{\partial f}{\partial y}(P_{ij})\right)^2 + 1}\, \Delta x\, \Delta y$$

$$= \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, \mathrm{d}A$$

Surface Area. The *area* of the surface S with equation $z = f(x, y) : (x, y) \in D$ where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous is

$$A(S) = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, \mathrm{d}A. \tag{4.3}$$

Question 4.14. Find the *surface area* of the part of the surface $z = x^2 + 2y$ that lies above the *triangular region* T in the xy-plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$.

ANSWER. The region T is given by $T = [0, 1] \times [0, x]$ and using (4.3) with $f(x, y) = x^2 + 2y$ we calculate

$$A = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, \mathrm{d}A = \iint_T \sqrt{(2x)^2 + (2)^2 + 1}\, \mathrm{d}A$$

$$= \int_0^1 \int_0^x \sqrt{4x^2+5}\, \mathrm{d}y\, \mathrm{d}x = \cdots = \frac{1}{12}(27 - 5\sqrt{5}).$$

◆

Question 4.15. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies *under* the plane $z = 9$ (i.e. $z \leq 9$).

ANSWER. The plane $z = 9$ intersects the paraboloid in the circle $x^2 + y^2 = 9$. Therefore our integrating region is

$$D = \{\text{disc of radius 4 centered at the origin}\} = \left\{(x, y) : x^2 + y^2 \leq 9\right\}.$$

Using the definition of surface area we write

$$A = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, \mathrm{d}A = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, \mathrm{d}A.$$

Converting to polar we obtain

$$\iint_D \sqrt{1 + (2x)^2 + (2y^2)}\, \mathrm{d}A$$
$$= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, \mathrm{d}r\, \mathrm{d}\theta = \cdots = \frac{\pi}{6}\left(37\sqrt{37} - 1\right).$$

◆

Exercise 4.2. By doing an arc length calculation, prove that the equation for the circumference of the circle of radius r is $2\pi r$.

Exercise 4.3. Prove the surface area of the sphere of radius r is given by $4\pi r^2$.

5 Change of Variables during Integration

We investigate how changing our coordinate system (i.e. the way we reference points in space) effects integrals.

5.1 Rectangles in the Cartesian Coordinate

The 'easiest' region to define in Cartesian is a rectangle — this is why Cartesian is sometimes called rectangular.

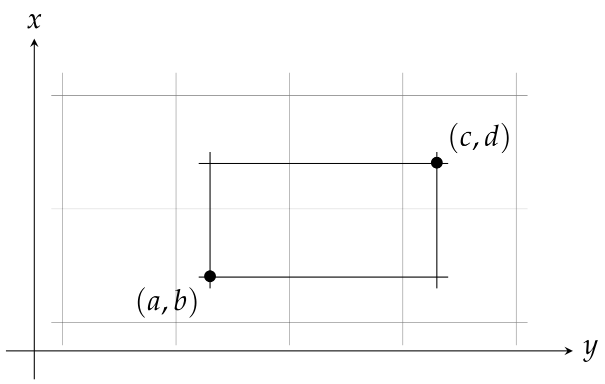
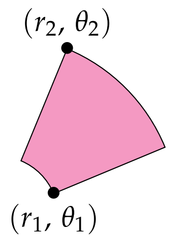


Figure 5.1: The rectangular region $R = [a, b] \times [c, d]$.

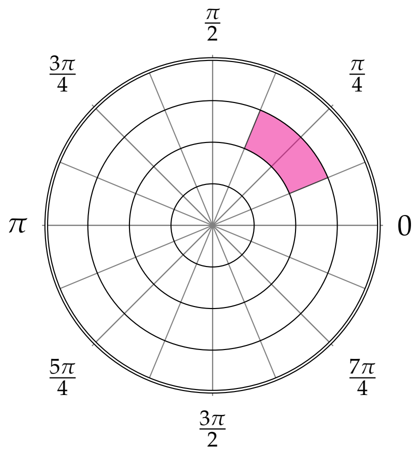
The 'easiest' region to define in the polar plane is an annulus.

Definition 5.1 (Annulus). An *annulus* or *polar rectangle* is the region $R = [r_1, \theta_1] \times [r_2, \theta_2]$:

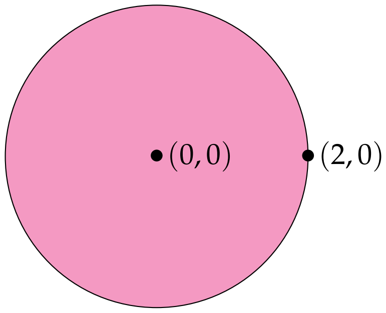


Example 5.2. The "rectangular" region in the polar plane given by

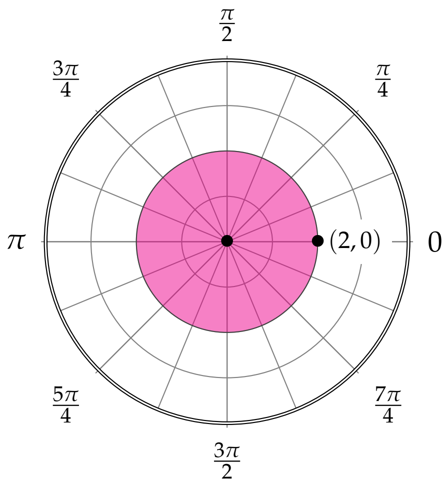
$$R = [1,2] \times \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] = \left\{(r,\theta) : 1 \leq r \leq 2, \frac{\pi}{8} \leq \theta \leq \frac{3\pi}{8}\right\}.$$



Question 5.3. Consider the figure below (labelled in Cartesian). Is it a polar rectangle?



ANSWER. Yes, the one given by $R = [0,2] \times [0,2\pi]$



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5.2 Riemann Sums over Annuli

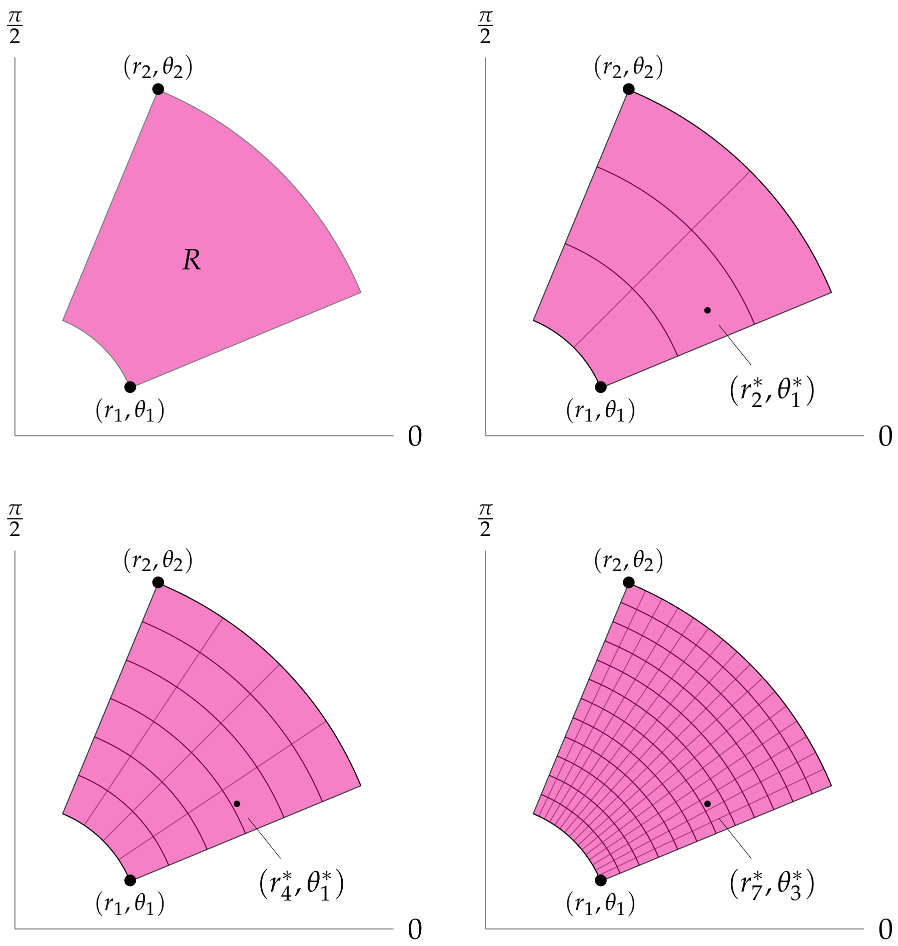
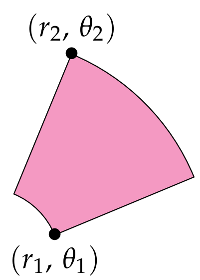
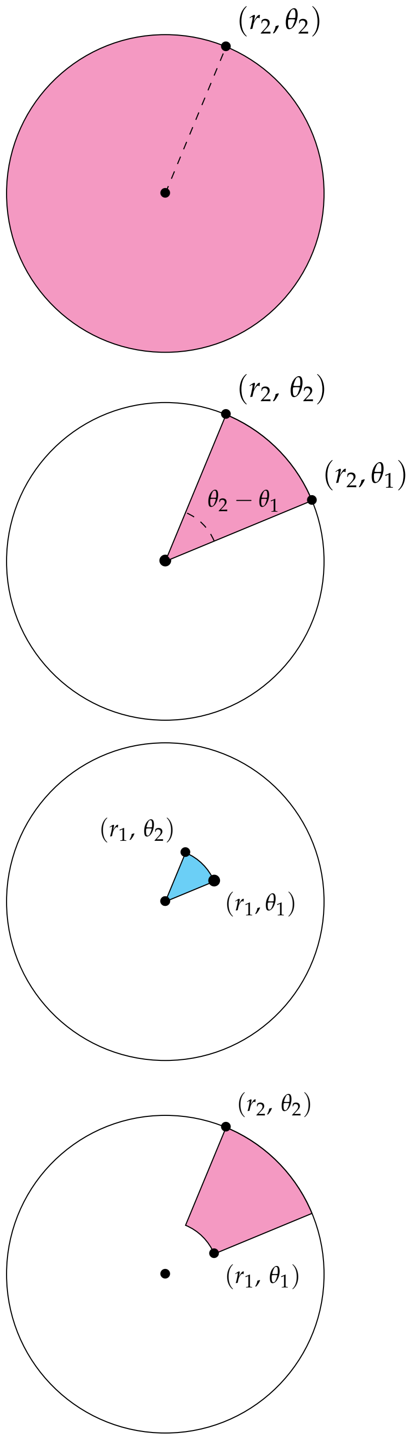


Figure 5.2: Dividing the annuli into 1×1, 3×2, 6×4 and 12×16 sub-annuli and taking a *sample point*.

Question 5.4. What is the area of the annulus below?



ANSWER. See Figure 5.3. ◆



$$A = \pi r_2^2$$

$$A_2 = \left(\frac{\theta_2 - \theta_1}{2\pi}\right) A = \frac{\theta_2 - \theta_1}{2} r_2^2$$

$$A_1 = \left(\frac{\theta_2 - \theta_1}{2\pi}\right) \pi r_1^2 = \frac{\theta_2 - \theta_1}{2} r_1^2$$

$$A_2 - A_1 = \frac{\theta_2 - \theta_1}{2}(r_2^2 - r_1^2)$$

Figure 5.3: For Question 5.4 — finding the area of an annulus.

Proposition 5.5. Let $r^* \in [r_1, r_2]$, $\Delta r = r_2 - r_1$, and $\Delta\theta = \theta_2 - \theta_1$ then

$$\text{Area}([r_1, r_2] \times [\theta_1, \theta_2]) \approx r^* \Delta r \Delta\theta.$$

PROOF. Consider,

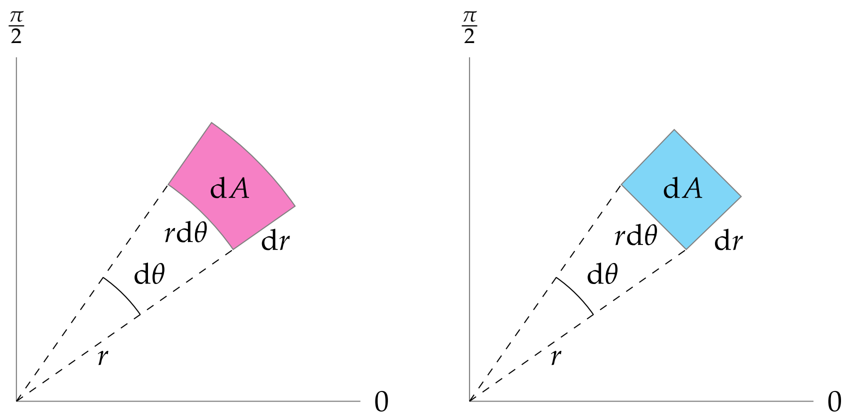
$$\frac{\theta_2 - \theta_1}{2}(r_2^2 - r_1^2) = (\theta_2 - \theta_1)(r_2 - r_1)\frac{(r_2 + r_1)}{2}$$

and notice

$$\frac{r_2 + r_1}{2}, r^* \in [r_1, r_2] \implies \frac{r_2 + r_1}{2} - r^* \leq \Delta r.$$

Thereby $\text{Area}([r_1, r_2] \times [\theta_1, \theta_2]) \approx r^*(r_2 - r_1)(\theta_2 - \theta_1)$ with the *approximation tending to exact* as $\Delta r \to 0$. ■

ALTERNATE PROOF. Recall we write $\mathrm{d}x$ when Δx is very small (i.e. $\Delta x \to 0$). In this limit the annulus becomes a rectangle:



($r\mathrm{d}\theta = \frac{\mathrm{d}\theta}{2\pi} \cdot 2\pi r$ is a proportion of the total circumference of the circle.) Thus $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$. ■

To estimate the volume bounded by $f(r, \theta)$ we need to *sum the volumes of annuli-based prisms*. Supposing the region has been subdivided into nm sub-annuli so that

$$\Delta r = \frac{r_2 - r_1}{n} \qquad \Delta\theta = \frac{\theta_2 - \theta_1}{m}$$

then the *volume of the prism* at the sample point (r^*, θ^*) is

$$r^* \Delta r \Delta\theta \cdot f(r^*, \theta^*).$$

Definition 5.6 (Polar Double Integral). Let $g(r, \theta) : \mathbb{R} \times [0, 2\pi] \to \mathbb{R}$ be a real polar function with

$$R = [r_1, r_2] \times [\theta_1, \theta_2] \subseteq \text{dom}(g).$$

Assume further that R is divided into nm-many sub-annuli with (r_i^*, θ_j^*) in the (i,j)th sub-annuli. Then

$$\iint\limits_R g(r,\theta)\, \mathrm{d}A = \int\limits_{\theta_1}^{\theta_2}\int\limits_{r_1}^{r_2} g(r,\theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta = \lim_{n,m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} g(r_i^*, \theta_j^*)\, r^*\Delta r\Delta\theta.$$

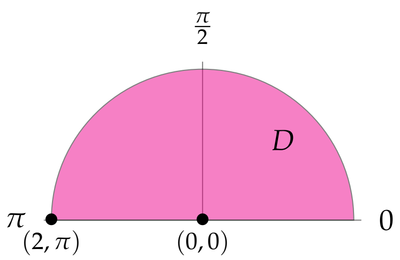
Proposition 5.7. If $f : \mathbb{R}^2 \to \mathbb{R}$ is Cartesian function that is continuous on the polar rectangle

$$R = [a,b] \times [\alpha, \beta]$$

with $0 \le a \le b$ and $0 \le \beta - \alpha \le 2\pi$, then

$$\iint\limits_R f(x,y)\, \mathrm{d}A = \int\limits_{\alpha}^{\beta}\int\limits_{a}^{b} f(r\cos\theta, r\sin\theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta. \tag{5.1}$$

Question 5.8. Let $D = \left\{(x,y) : x^2 + y^2 \le 4\right\}$



and evaluate $\iint_D (1 + x^2 + y^2)\, \mathrm{d}A$.

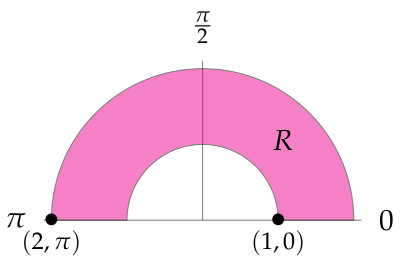
ANSWER.

$$\begin{aligned}
&\iint_D (1 + x^2 + y^2)\, \mathrm{d}A \\
&\quad = \int_0^{2\pi}\int_0^2 (1 + r^2)\, r\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi}\int_0^2 (r + r^3)\, \mathrm{d}r\, \mathrm{d}\theta \\
&\quad = \int_0^{2\pi}\left[\frac{r^2}{2} + \frac{r^4}{4}\right]_0^2 \mathrm{d}\theta = \int_0^{2\pi} 6\, \mathrm{d}\theta = 12\pi
\end{aligned}$$

◆

Question 5.9. Let R be the region in the upper half-plane (i.e. $y \ge 0$)

bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.



Evaluate $\iint_R (3x+4y^2)\,\mathrm{d}A$.

ANSWER. We have $R=[1,2]\times[0,\pi]$ and thus

$$\begin{aligned}
&\iint_R (3x+4y^2)\,\mathrm{d}A \\
&\quad= \int_0^\pi \int_1^2 3(r\cos\theta)+4(r^2\sin^2\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta && \text{(convert to polar)} \\
&\quad= \int_0^\pi \Big[r^3\cos\theta + r^4\sin^2\theta\Big]_1^2\,\mathrm{d}\theta \\
&\quad= \int_0^\pi (8\cos\theta+16\sin^2\theta)-(\cos\theta+\sin^2\theta)\,\mathrm{d}\theta \\
&\quad= \int_0^\pi 7\cos\theta+15\sin^2\theta\,\mathrm{d}\theta \\
&\quad= \int_6^\pi 7\cos\theta+\frac{15}{2}(1-\cos 2\theta)\,\mathrm{d}\theta && \text{(double angle formula)} \\
&\quad= \left[7\sin\theta+\frac{15}{2}\theta-\frac{15}{4}\sin 2\theta\right]_0^\pi = \frac{15}{2}\pi
\end{aligned}$$

◆

Question 5.10. Compute the *volume* of the hemisphere (upper half of the sphere)

$$H=\{(x,y,z) : x^2+y^2+z^2\le a^2,\ z\ge 0\}$$

over the disc (interior of a circle, including boundary)

$$D=\{(x,y) : x^2+y^2\le a^2\}.$$

Note since the volume of a sphere is $\frac{4}{3}\pi r^3$ we know the answer is $\frac{2}{3}\pi a^3$.

ANSWER. We can solve $x^2+y^2+z^2\le a^2$ for z to obtain $z\le\pm\sqrt{a^2-x^2-y^2}$ which combined with $z\ge 0$ means $z\le\sqrt{a^2-x^2-y^2}$. Also the interior of the radius-a disc in polar is $D=[0,a]\times[0,2\pi]$. Finally

$$\begin{aligned}
&\iint_D \sqrt{a^2-x^2-y^2}\,\mathrm{d}A \\
&\quad= \int_0^{2\pi}\int_0^a \sqrt{a^2-r^2\cos^2\theta-r^2\sin^2\theta}\; r\,\mathrm{d}r\,\mathrm{d}\theta && \text{(convert to polar)}
\end{aligned}$$

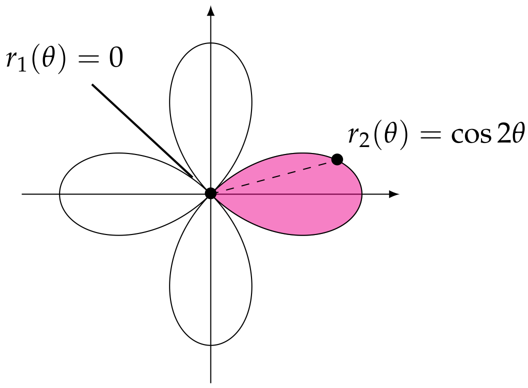
$$= \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}\, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{(a+r)^2 \sqrt{a^2 - r^2}}{3} \right]_0^a d\theta \qquad \text{(by parts with table lookup)}$$

$$= \int_0^{2\pi} \frac{a^3}{3} \, d\theta = \frac{2}{3}\pi a^3.$$

As expected. ◆

Question 5.11. Find the area enclosed by one loop (i.e. $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$) of the four-leaved rose $r = \cos 2\theta$.



For each θ we take all values of $r \in [0, \cos 2\theta]$.

ANSWER. Our region D is given by

$$D = [0, \cos 2\theta] \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

and to find the *bounded area* we *integrate over the unit function*

$$\iint_D 1 \, dA$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\cos 2\theta} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

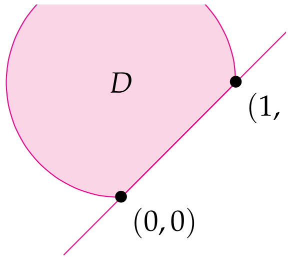
$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 4\theta) \, d\theta \qquad \text{double angle formula}$$

$$= \frac{1}{4} \left[\theta + \frac{1}{4} \sin 4\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8}.$$

◆

Question 5.12. Let D be the region *inside* the circle $x^2 + (y-1)^2 = 1$

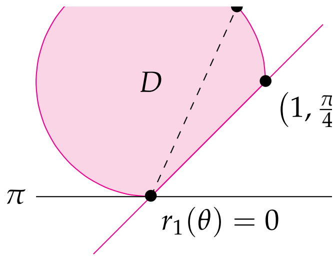
bounded *below* by the line $y = x$.



Evaluate $\iint_D x \, \mathrm{d}A$.

ANSWER. Converting $x^2 + y^2 - 2y = 0$ to polar we have

$$r^2 \sin^2\theta + r^2\cos^2\theta - 2\cos\theta = 0 \implies r = 2\sin\theta$$



And thus $D = [0, 2\sin\theta] \times \left[\frac{\pi}{4}, \pi\right]$.

$$\begin{aligned}
&\iint_D x \, \mathrm{d}A \\
&= \int_{\frac{\pi}{4}}^{\pi} \int_0^{2\sin\theta} r\sin\theta \, r \, \mathrm{d}r \, \mathrm{d}\theta = \int_{\frac{\pi}{4}}^{\pi} \left[\frac{r^3}{3}\sin\theta\right]_0^{2\sin\theta} \mathrm{d}\theta \\
&= \frac{8}{3}\int_{\frac{\pi}{4}}^{\pi} \sin^4\theta \, \mathrm{d}\theta \\
&= \frac{8}{3}\left[-\frac{1}{4}\sin^3\theta\cos\theta - \frac{3}{8}\cos\theta\sin\theta + \frac{3}{8}\theta\right]_{\frac{\pi}{4}}^{\pi} && \text{(table lookup)} \\
&= \frac{3}{4}\pi + \frac{2}{3}.
\end{aligned}$$

◆

5.3 *Change of Variables*

Consider converting from rectangular to polar. This *change of variables* was done by *transforming* the real plane into the polar one via

$$\mathbb{R}^2 \to \mathbb{R} \times [0, 2\pi)$$

$$(x, y) \mapsto \left(\sqrt{x^2 + y^2}, \arctan \frac{y}{x}\right).$$

We have already seen performing the transform can help make integrals easier. Let us investigate what other transforms are possible.

Relation. A *relation/mapping* T between two sets A and B is given by

$$T : A \to B$$
$$a \mapsto b.$$

The *image* of A on the under the transformation is

$$T(A) = \{T(a) : a \in A\} \subseteq B.$$

One-to-one. T is *one-to-one* when

$$T(a_1) = T(a_2) \iff a_1 = a_2.$$

A *function* is one-to-one if it passes *both* the *vertical line test* and the *horizontal line test*.

Proposition 5.13. If $T : A \to B$ is one-to-one then T is *invertible*. That is, there is $T^{-1} : B \to A$ such that

$$T(a) = b \iff T^{-1}(b) = a.$$

Note T^{-1} is also one-to-one.

PROOF. Omitted. ■

Question 5.14. Let $T(x) = \sqrt{x}$. Is $T(x)$ one-to-one? If yes, what is T's inverse?

ANSWER. Yes. $T^{-1}(y) = y^2$ where $T^{-1} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$. ◆

Question 5.15. Let $T(x) = x^2$. Is $T(x)$ one-to-one? If yes, what is T's inverse?

ANSWER. No. $T(1) = T(-1) = 1$. Notice $T^{-1}(1) = 1$ *or* -1. ◆

Transform. A *transformation* of $\mathbb{R}^2$ is any mapping

$$T : U \times V \to \mathbb{R} \times \mathbb{R}$$
$$(u, v) \mapsto (x, y)$$

such that T is one-to-one.

Notation. When $T(u,v) = (x,y)$ we write

$$x = x(u,v) \qquad y = y(u,v)$$

so that $T(u,v) = (x(u,v), y(u,v))$.

Example 5.16. The transformation for converting *from* polar into $\mathbb{R}^2$ is given by

$$\begin{aligned} T : \mathbb{R} \times [0, 2\pi) &\to \mathbb{R} \times \mathbb{R} \\ (r, \theta) &\mapsto (r\cos\theta,\, r\sin\theta) \end{aligned}$$

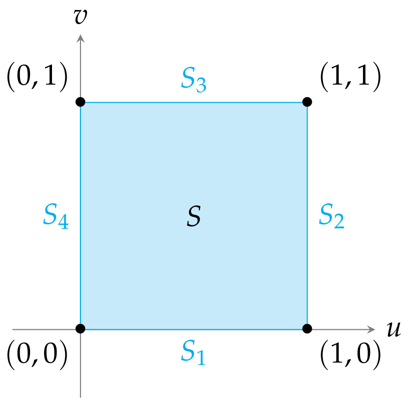
and the *inverse* transform is given by

$$\begin{aligned} T^{-1} : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \times [0, 2\pi) \\ (x, y) &\mapsto \left(\sqrt{x^2 + y^2},\, \arctan\frac{y}{x}\right). \end{aligned}$$

Question 5.17. What is the image of the set $S = [0,1] \times [0,1] \subseteq U \times V$ under the transformation defined by

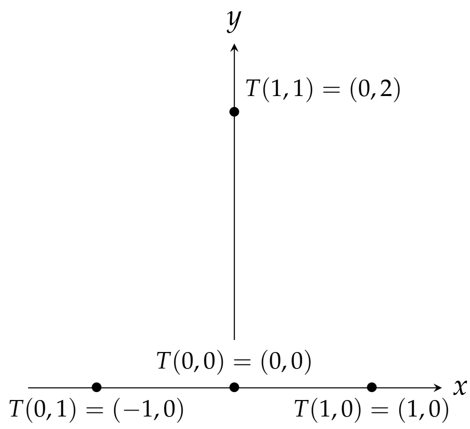
$$\begin{aligned} T : U \times V &\to \mathbb{R}^2 \\ (u, v) &\mapsto (u^2 - v^2,\, 2uv). \end{aligned}$$

ANSWER. In the uv plane the S-region looks like:



Where $S_1, \ldots, S_4$ are four lines comprising the *boundary* of S.

Recalling $T(u,v) = (u^2 - v^2, 2uv)$ we can map the *endpoints* to new endpoints in the xy-plane.



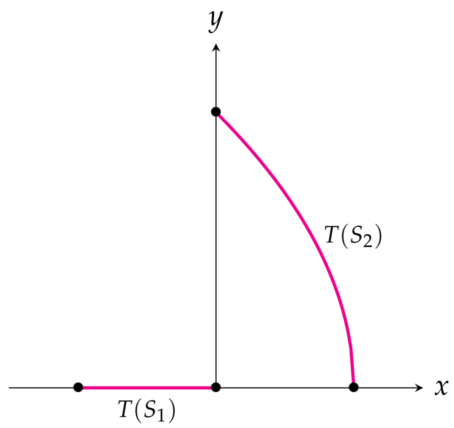
Now we need the images of $S_1, \dots, S_4$ to find the new boundaries: $S_1 = \{(u,0) : u \in [0,1]\}$ implies

$$\begin{aligned} T(S_1) &= \{T(u,0) : u \in [0,1]\} \\ &= \{(u^2,0) : u \in [0,1]\} \\ &= \{(x,0) : x \in [0,1]\}. \end{aligned}$$

$S_2 = \{(1,v) : v \in [0,1]\}$ implies

$$\begin{aligned} T(S_2) &= \{T(1,v) : v \in [0,1]\} \\ &= \{(1-v^2, 2v) : v \in [0,1]\} \\ &= \left\{\left(x, 1-\frac{y^2}{4}\right) : x \in [0,1]\right\}. \end{aligned}$$

where $x = 1 - v^2 \wedge y = 2v \implies x = 1 - \frac{y^2}{4}$



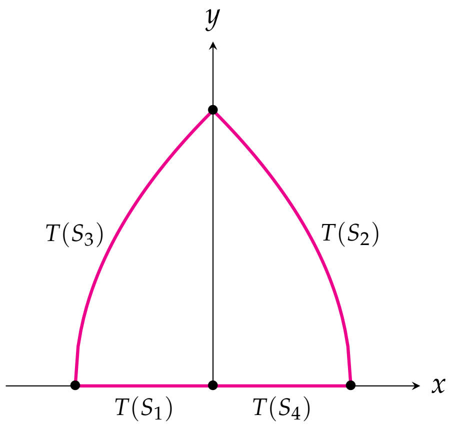
$S_3 = \{(u,1) : u \in [0,1]\}$ implies

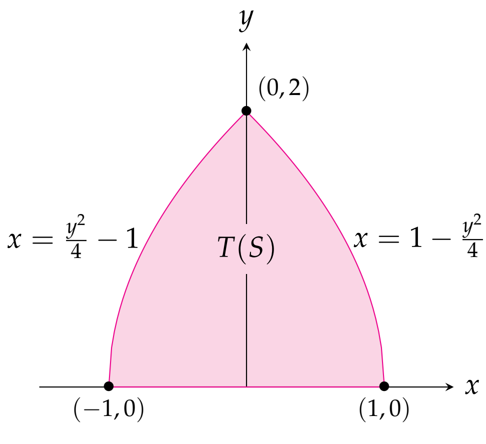
$$T(S_3) = \{T(u,1) : u \in [0,1]\}$$

$$= \{(u^2 - 1, 2u) : u \in [0,1]\}$$
$$= \left\{\left(x, \frac{y^2}{4} - 1\right) : x \in [0,1]\right\}.$$

where $x = u^2 - 1 \wedge y = 2u \implies x = \frac{y^2}{4} - 1$ and $S_4 = \{(0,v) : v \in [0,1]\}$ implies

$$T(S_2) = \{T(0,v) : v \in [0,1]\}$$
$$= \{(-v^2, 0) : v \in [0,1]\}$$
$$= \{(x,0) : \in [-1,0]\}.$$

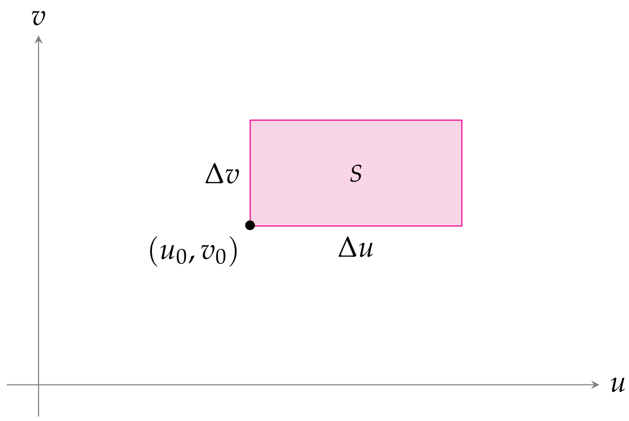




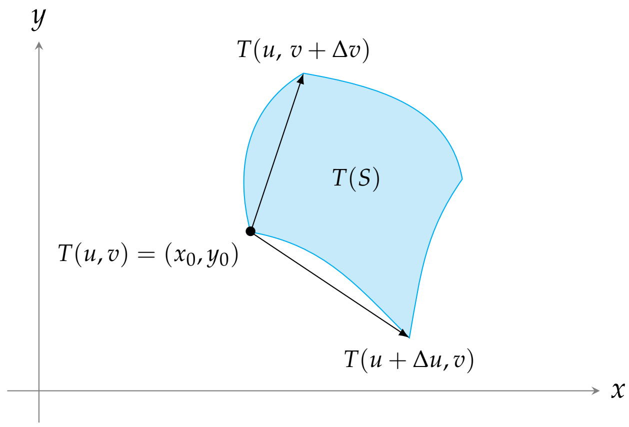
◆

Now let us see how a change of variables affects a double integral. Basically we need to investigate how the base-area of our prisms are effected.

Consider our usual *rectangular* base whose $\text{Area}(S) = \Delta u \Delta v$.



We estimate the area of non-rectangular regions using tangent vectors.



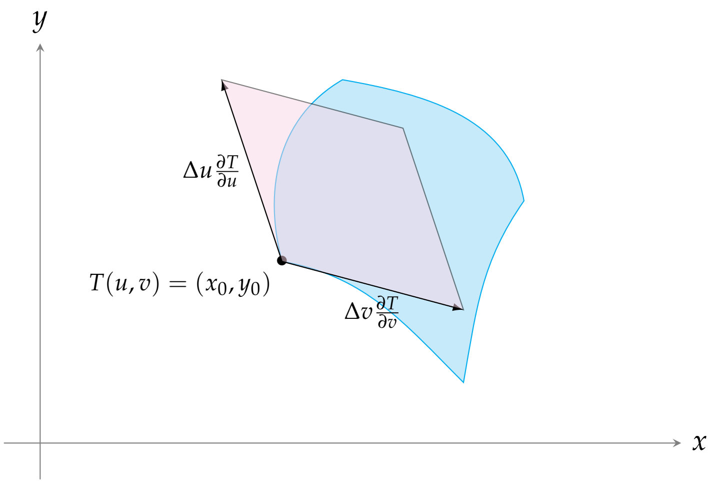
Notice

$$\frac{\partial T}{\partial u} = \lim_{\Delta u \to 0} \frac{T(u_0 + \Delta u,\, v_0) - T(u_0, v_0)}{\Delta u}$$
$$\implies \Delta u \frac{\partial T}{\partial u} \approx T(u_0 + \Delta u,\, v_0) - T(u_0, v_0)$$

and thereby the area of the parallelogram is given by the length of the cross product

$$\left| \Delta u \frac{\partial T}{\partial u} \times \Delta v \frac{\partial T}{\partial v} \right| = \left| \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \right| \Delta u \Delta v$$

and is a suitable approximation of Area$(T(S))$.



Computing the cross product we obtain

$$\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} = \left\langle \frac{\partial x(u,v)}{\partial u}, \frac{\partial y(u,v)}{\partial u}, 0 \right\rangle \times \left\langle \frac{\partial x(u,v)}{\partial u}, \frac{\partial y(u,v)}{\partial v}, 0 \right\rangle$$
$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \boldsymbol{k}$$

Definition 5.18 (Jacobian). The *Jacobian* of the transformation $T(u, v) = (x(u,v), y(u,v))$ is

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Example 5.19. For *polar* the Jacobian is extracted from $T(r,\theta) = (r\cos\theta, y\sin\theta)$:

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial r} \\ \frac{\partial r\cos\theta}{\partial \theta} & \frac{\partial r\sin\theta}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

With this notation we have

$$\text{Area}(T(S)) = \left| \left[\frac{\partial(x,y)}{\partial(u,v)}\right]_{(u,v)=(u_0,v_0)} \right| \Delta u \Delta v.$$

Proposition 5.20. Let R be a rectangular region in $\mathbb{R}^2$, T a transformation from $U \times V \to \mathbb{R}^2$, and $S \subseteq U$ such that $T(S) = R$.

$$\iint_R f(x,y)\, \mathrm{d}A = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\, \mathrm{d}v. \qquad (5.2)$$

PROOF.

$$\begin{aligned} \iint_R f(x,y)\, \mathrm{d}A &\approx \sum_i^m \sum_j^n f(x_i, y_i)\, \Delta A \\ &\approx \sum_i^m \sum_j^n f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u\, \Delta v \\ &\approx \iint_S f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\, \mathrm{d}v \end{aligned}$$

■

Let us confirm this proposition for *polar* which is given by T with

$$x(r,\theta) = r\cos\theta \qquad\qquad y(r,\theta) = r\sin\theta.$$

Our proposition says

$$\begin{aligned} \iint_R f(x,y)\, \mathrm{d}x\mathrm{d}y &= \iint_{T^{-1}(R)} f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| \mathrm{d}r\mathrm{d}\theta \\ &= \iint_{T^{-1}(R)} f(r\cos\theta, r\sin\theta)\, r\mathrm{d}r\mathrm{d}\theta \end{aligned}$$

which checks out with what we developed before. Note $T^{-1}(R)$ is the image of R under the *inverse* of T — sometimes called the *preimage*.

Question 5.21. Use the transformation $T(u,v) = (u^2 - v^2, 2uv)$ to evaluate

$$\int_R y\, \mathrm{d}A$$

where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$

ANSWER. From earlier this lecture, we know that $T([0,1] \times [0,1]) = R$. The *Jacobian* of T is given by

$$\begin{vmatrix} \frac{\partial u^2 - v^2}{\partial u} & \frac{\partial 2uv}{\partial u} \\ \frac{\partial u^2 - v^2}{\partial v} & \frac{\partial 2uv}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 4u^2 - 4v^2.$$

Thereby

$$\iint_R y\, \mathrm{d}A = \int_0^1 \int_0^1 (2uv)\, 4(u^2 + v^2)\, \mathrm{d}u\, \mathrm{d}v$$

$$= 8\int_0^1\int_0^1 u^3v + uv^3 \, \mathrm{d}u\,\mathrm{d}v = 8\int_0^1 \left[\frac{1}{4}u^4 + \frac{1}{2}u^2v^3\right]_0^1 \mathrm{d}v$$
$$= 8\int_0^1 2v + 4v^3 \, \mathrm{d}v = \left[v^2 + v^4\right]_0^1 = 2.$$

◆

Question 5.22. Let R be the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$ as illustrated in Figure 5.5. Evaluate $\iint_R \mathrm{e}^{(x+y)/(x-y)} \, \mathrm{d}A$.

ANSWER. Our intuition suggests that letting

$$u = x + y \qquad\qquad v = x - y$$

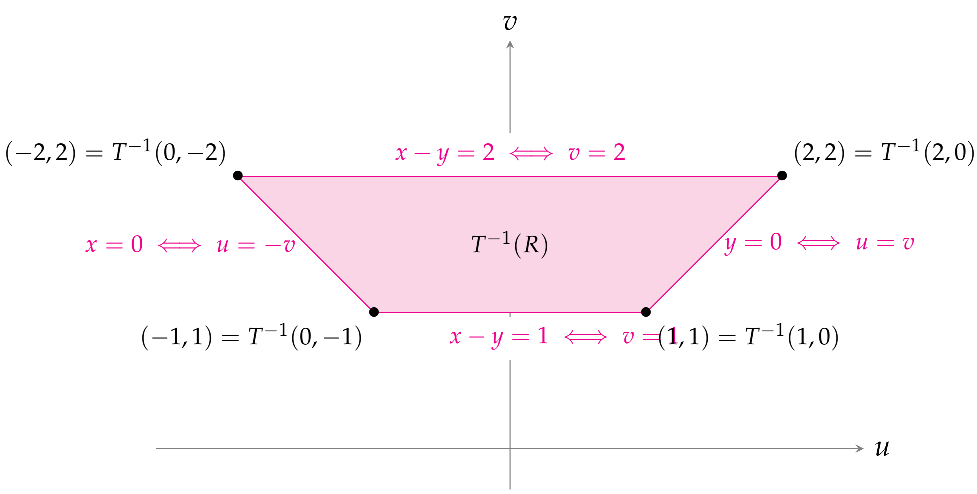
may help integrate $\mathrm{e}^{(x+y)/(x-y)}$ somehow.

Notice $u + v = 2x$ and $u - v = 2y$ and thus

$$x(u,v) = \frac{u+v}{2} \qquad\qquad y(u,v) = \frac{u-v}{2}.$$

It is important to note here that we are *not* doing a change of variables to make the integrating region simpler, but to make the *integrand* simpler.

Next we determine our new region $T^{-1}(R)$ by finding the image of R's endpoints under the transformation



Note $x = 0 \iff \frac{1}{2}(u+v) = 0 \iff u = -v$ and $y = 0 \iff \frac{1}{2}(u-v) = 0 \iff u = v$.

The Jacobian of T is

$$\begin{vmatrix} \frac{\partial}{\partial u}\frac{u+v}{2} & \frac{\partial}{\partial u}\frac{u-v}{2} \\ \frac{\partial}{\partial v}\frac{u+v}{2} & \frac{\partial}{\partial v}\frac{u-v}{2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = -\frac{1}{2}.$$

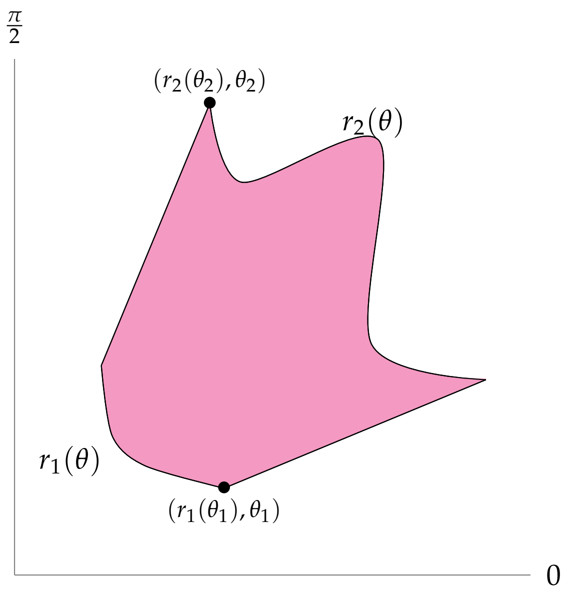


Figure 5.4: At $\theta \in [\theta_1, \theta_2]$ we have $r \in [r_1(\theta), r_2(\theta)]$.

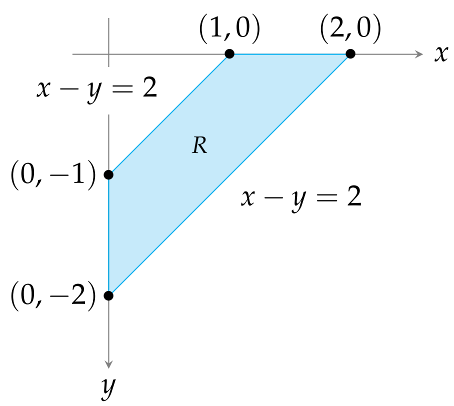


Figure 5.5: The trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

Thus

$$\iint_R e^{(x+y)/(x-y)}\, dA$$

$$= \iint_{T^{-1}(R)} e^{u/v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du\, dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v} \left| -\frac{1}{2} \right| \, du\, dv \; = \frac{1}{2} \int_1^2 \left[v e^{u/v} \right]_{-v}^{v} \, dv$$

$$= \frac{1}{2} \int_1^2 v e^1 - \frac{v}{e^1} \, dv \; = \frac{1}{2} \left[\frac{v^2}{2} \left(e^1 - \frac{1}{e^1} \right) \right]_1^2$$

$$= \frac{1}{2} \frac{2^2 - 1^2}{2} \left(e^1 - \frac{1}{e^1} \right) = \frac{3}{4} \left(e^1 - \frac{1}{e^1} \right).$$

◆

6 *Triple Integrals*

Just as we defined *single* and *double* integrals over *lines* and *rectangles*, we can define *triple* integrals over *rectangular prisms*.

Definition 6.1 (Rectangular Prism)**.** Let $a, b, c, d, r, s \in \mathbb{R}$ such that $a \leq b$, $c \leq d$, and $r \leq s$ then $B = [a, b] \times [c, d] \times [r, s]$ is a *rectangular prism* or *box*.

To construct the *triple Riemann sum* we again just break B into ℓmn many sub-prisms.

Definition 6.2 (Triple Integral)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a real function with $[a, b] \times [c, d] \times [r, s] \subseteq \text{dom}(f)$. Assume further that $R = [a, b] \times [c, d] \times [r, s]$ is divided into ℓmn-many equally sized sub-rectangles with (x_i^*, y_i^*, z_k^*) in the (i, j, k)th sub-rectangle. Then

$$\iiint\limits_R f(x)\ \mathrm{d}V = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{\ell} f(x_i^*, y_j^*, z_k^*)\ \Delta x\, \Delta y\, \Delta z.$$

Fubini's Theorem. If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$ then

$$\iiint\limits_B f(x, y, z)\ \mathrm{d}V = \int\limits_r^s \int\limits_c^d \int\limits_a^b f(x, y, z)\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

where the order of $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ may be permuted along with their bounds.

Question 6.3. Let $B = [0, 1] \times [-1, 2] \times [0, 3]$ and evaluate $\iint_B xyz^2 \mathrm{d}V$.

ANSWER.

$$\iint_B xyz^2 \mathrm{d}V = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} \mathrm{d}y\, \mathrm{d}z$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz = \int_0^3 \left[\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz = \int_0^3 \frac{3z^2}{4}\, dz = \left[\frac{z^3}{4}\right]_{z=0}^{z=3} = \frac{27}{4}$$

◆

Question 6.4. Let $B = [0,1] \times [0,3] \times [1,2]$ and evaluate $\iiint_B 6xze^{yz}\, dV$.

ANSWER.

$$\iiint_B 6xze^{yz}$$
$$= \int_0^1 \int_1^2 \int_0^3 6xze^{yz}\, dy\, dz\, dx = \int_0^1 \int_1^2 \left[6xe^{yz}\right]_{y=0}^{y=3} dz\, dx$$
$$= \int_0^1 \int_1^2 6xe^{3z} - 1\, dz\, dx = \int_0^1 \left[2x\left(e^{3z} - 3z\right)\right]_{z=1}^{z=2} dx$$
$$= \int_0^1 \left(e^6 - e^3 - 3\right) 2x\, dx = \left[\left(e^6 - e^3 - 3\right) x^2\right]_{x=0}^{x=1}$$
$$= e^6 - e^3 - 3$$

◆

To investigate what other regions we can triple integrate over consider rewriting the *triple* integral as a *double* integral over the region

$[a,b] \times [c,d]$	$[a,b] \times [y_0(a), y_1(b)]$	$[x_0(c), x_1(d)] \times [c,d]$
Rectangular	Type I	Type II

In *any* of these cases C need only be of the form $C = [z_0(x,y), z_1(x,y)]$ for $z_0, z_1 : \mathbb{R}^2 \to \mathbb{R}$.

Proposition 6.5. Suppose $E = D \times [z_0(x,y), z_1(x,y)]$ where D is a Type 1 region. Namely

$$E_{xyz} = [a,\, b] \times [y_0(x),\, y_1(x)] \times [z_0(x,y),\, z_1(x,y)]$$

then

$$\iiint_{E_{xyz}} f(x,y,z)\, dV = \int_a^b \int_{y_0(x)}^{y_1(x)} \int_{z_0(x,y)}^{z_1(x,y)} f(x,y,z)\, dz\, dy\, dx$$

(This proposition is true for all permutations of x, y, z as well.)

Question 6.6. Evaluate $\iiint_E z\, dV$ for

$$E = \{\text{Tetrahedron bounded by } x = 0,\, y = 0,\, z = 0 \text{ and } x + y + z = 1\}.$$

ANSWER. We must first determine what values x, y, and z can obtain (in that arbitrary order). The minimum value for each is 0 whereas the maximum values are dependent on the order:

$$x \in [0, 1] \implies y \in [0, 1-x] \implies z \in [0, 1-x-y]$$

and thus

$$E_{xyz} = [0,1] \times [0, 1-x] \times [0, 1-x-y]$$

All that remains is the (routine) evaluation of the iterated integral

$$\iiint\limits_{E_{xyz}} z \, \mathrm{d}V$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} \mathrm{d}y \, \mathrm{d}x$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{2} \int_0^1 \left[-\frac{(1-x-y)^3}{3}\right]_{y=0}^{y=1-x} \mathrm{d}x$$

$$= \frac{1}{6} \int_0^1 (1-x)^3 \, \mathrm{d}x = \frac{1}{6} \left[-\frac{(1-x)^4}{4}\right]_0^1 = \frac{1}{24}$$

◆

Question 6.7. What other representations of $E = \{$ Tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1.$ $\}$ are there?

ANSWER. The *six* representations are

$$E_{xyz} = [0, 1] \times [0, 1-x] \times [0, 1-x-y]$$
$$E_{xzy} = [0, 1] \times [0, 1-x] \times [0, 1-x-z]$$
$$E_{yxz} = [0, 1] \times [0, 1-y] \times [0, 1-y-x]$$
$$E_{yzx} = [0, 1] \times [0, 1-y] \times [0, 1-y-z]$$
$$E_{zxy} = [0, 1] \times [0, 1-z] \times [0, 1-z-x]$$
$$E_{zyx} = [0, 1] \times [0, 1-z] \times [0, 1-z-y]$$

where E_{yxz}, for instance, denotes the order of integration is $\mathrm{d}y\mathrm{d}x\mathrm{d}z$. ◆

Question 6.8. Let E be the region bounded by the

1. paraboloid $z = x^2 + y^2$,
2. plane $z = 1$,
3. xy-plane $y = 0$, and
4. yz-plane $x = 0$.

and evaluate $\iiint_E xyz \, \mathrm{d}V$.

ANSWER. To express the region E notice that the minimum value for x and

y is 0 which means $z = 0^2 + 0^2 = 0$ is z's minimum. Thus

$$z \in [0,1] \implies y \in [0, \sqrt{z}\,] \implies x \in \left[0, \sqrt{z - y^2}\right].$$

All that remains is to perform the iterated integral with

$$E_{zyx} = [0,\ 1] \times [0,\ \sqrt{z}] \times [0,\ \sqrt{z - x^2}].$$

$$\iiint\limits_{E_{zyx}} xyz \, \mathrm{d}V$$

$$= \int_0^1 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} xyz \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_0^1 \int_0^{\sqrt{z}} \left[\frac{1}{2}x^2 yz\right]_{x=0}^{x=\sqrt{z-y^2}} \mathrm{d}y \, \mathrm{d}z$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{z}} (z - y^2)yz \, \mathrm{d}y \, \mathrm{d}z = \frac{1}{2}\int_0^1 \left[\frac{1}{2}y^2 z^2 - \frac{1}{4}y^4 z\right]_{y=0}^{y=\sqrt{z}} \mathrm{d}z$$

$$= \frac{1}{8}\int_0^1 z^3 \, \mathrm{d}z = \frac{1}{8}\left[\frac{1}{4}z^4\right]_0^1 = \frac{1}{32}.$$

◆

Question 6.9. Let E be the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$ and evaluate $\iiint_E \sqrt{x^2 + z^2} \, \mathrm{d}V$.

ANSWER. We express our region E by noting that $y = x^2 + z^2$ can never be negative and thus $0 \le x^2 + z^2 \le 4 \implies x \in [-2, 2]$. Thereby we have $E_{xyz} = [-2,\ 2] \times [x^2,\ 4] \times [-\sqrt{y - x^2},\ \sqrt{y - x^2}]$.

$$\iiint\limits_{E_{xyz}} \sqrt{x^2 + z^2} \, \mathrm{d}V = \int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2} \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x.$$

Although this expression is correct, it wold be difficult to evaluate. Let us try a different representation for E — one that enables us to integrate by y *first*: $E_{xzy} = [-2, 2] \times \left[-\sqrt{4 - x^2},\ \sqrt{4 - x^2}\right] \times \left[x^2 + z^2,\ 4\right]$.

$$\iiint\limits_{E_{xzy}} \sqrt{x^2 + z^2} \, \mathrm{d}V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} \sqrt{x^2 + z^2} \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}x$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[y\sqrt{x^2 + z^2}\right]_{x^2+z^2}^{4} \mathrm{d}z \, \mathrm{d}x$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2} \, \mathrm{d}z \, \mathrm{d}x.$$

Let us convert to polar using $x \leftarrow r\cos\theta$ and $z \leftarrow r\sin\theta$. Note that here we have $z^2 = 4 - x^2$ for $x \in [-2, 2]$ which means our polar region is

$[0,2\pi] \times [0,2]$ and so

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2+z^2}\, \mathrm{d}z\, \mathrm{d}x$$
$$= \int_0^{2\pi} \int_0^2 (4-r^2) r\, r\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} \int_0^2 4r^2 - r^4\, \mathrm{d}r\, \mathrm{d}\theta$$
$$= \int_0^{2\pi} \left[\frac{4}{3}r^3 - \frac{1}{5}r^5\right]_0^2 \mathrm{d}\theta = \int_0^{2\pi} \frac{64}{15}\, \mathrm{d}\theta = \frac{128\pi}{15}.$$

◆

Question 6.10. Rewrite $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$ as an iterated integral with $V = \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y$.

ANSWER. The given region is $E_{xyz} = [0, 1] \times [0, x^2] \times [0, y]$ which implies

$$(0 \le x \le 1) \wedge (0 \le y \le x^2) \wedge (0 \le z \le y)$$
$$\implies (0 \le y \le x^2 \le 1) \wedge (\sqrt{y} \le x \le 1) \wedge (0 \le z \le y)$$
$$\implies y \in [0,1] \wedge z \in [0,y] \wedge x \in [\sqrt{y},1].$$

Thus $E_{yzx} = [0, 1] \times [0, y] \times [\sqrt{y}, 1]$ and thereby

$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

◆

6.1 Change of Variables in Triple Integrals

Definition 6.11 (Jacobian). Let T be a *transformation* which takes R in uvw-space to S in xyz-space via the maps

$$x = x(u,v,w) \qquad y = y(u,v,w) \qquad z = z(u,v,w)$$

then the *Jacobian* of T is defined to be

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Theorem 6.12. Suppose that a transformation is given by

$$T : U \times V \times W \to \mathbb{R}^3$$
$$(u, v, w) \mapsto (x(u, v, w), \ y(u, v, w)\, z(u, v, w))$$

Then $\iiint_R f(x, y, z)\, dV = \iiint_{T^{-1}(R)} f(T(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$

Let us apply this theorem for the *spherical* and *cylindrical* change of variables.

6.1.1 Cylindrical Change of Variables

Cylindrical. The transformation T given by

$$T : \mathbb{R} \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$$
$$(r,\ \theta,\ z) \mapsto (r\cos\theta,\ r\sin\theta,\ z)$$

converts *from cylindrical to rectangular*.

Question 6.13. What is the Jacobian for the transformation T which maps from cylindrical to rectangular?

ANSWER.

$$\frac{\partial(r\cos\theta,\ r\sin\theta,\ z)}{\partial(r,\ \theta,\ z)} = \begin{vmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} & \frac{\partial r\cos\theta}{\partial z} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial \theta} & \frac{\partial r\sin\theta}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= r\cos^2\theta + r\sin^2\theta = r$$

◆

Proposition 6.14. Let T be the transformation from cylindrical to rectangular. Then

$$\iiint_R f(x, y, z)\, dV = \iiint_{T^{-1}(R)} f(r\cos\theta,\ r\sin\theta,\ z)\, r\, dr\, d\theta\, dz.$$

Question 6.15. Let E be the region

1. *inside* the cylinder $x^2 + y^2 - 1$,
2. *below* the plane $z = 4$, and

3. *above* the paraboloid $z = 1 - x^2 - y^2$.

and evaluate $\iiint_E \sqrt{x^2+y^2}\,\mathrm{d}V$.

ANSWER. *Inside* the cylinder, *below* the plane, and *above* the paraboloid give

$$0 \le x^2 + y^2 \le 1 \qquad z \le 4 \qquad 1 - x^2 - y^2 \le z$$

Converting to cylindrical gives

$$0 \le r^2 \le 1 \qquad z \le 4 \qquad 1 - r^2 \le z$$

which simplifies to

$$0 \le r \le 1 \qquad 1 - r^2 \le z \le 4 \qquad \theta \text{ unrestricted.}$$

Thereby $E_{\theta r z} = [0, 2\pi) \times [0, 1] \times [1 - r^2, 4]$.

$$\begin{aligned}
&\iiint_E \sqrt{x^2+y^2}\,\mathrm{d}V \\
&= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}\; r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 r^2\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^1 \left[zr^2\right]_{1-r^2}^4\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^1 3r^2 + r^4\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\left[r^3 + \frac{r^5}{5}\right]_0^1 \mathrm{d}\theta \\
&= \int_0^{2\pi}\frac{6}{5}\,\mathrm{d}\theta = \frac{12}{5}\pi.
\end{aligned}$$

◆

Question 6.16. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

ANSWER. First we observe that

$$E_{xy} = [-2, 2] \times \left[-\sqrt{4-x^2},\ \sqrt{4-x^2}\right]$$

is the disc of radius 2 centered at the origin. Thereby

$$E_{r\theta} = [0, 2] \times [0, 2\pi).$$

We also have $z \in \left[\sqrt{x^2+y^2},\ 2\right]$ which gives us $z \in [r, 2]$ in cylindrical and thus

$$E_{r\theta z} = [0, 2] \times [0,\ 2\pi) \times [r, 2].$$

$$\begin{aligned}
&\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&\quad = \int_0^2\int_0^{2\pi}\int_r^2 (r^2\cos^2\theta + r^2\sin^2\theta)\; r\,\mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}r
\end{aligned}$$

$$= \int_0^2 \int_0^{2\pi} \int_r^2 r^3 \, dz \, d\theta \, dr = \int_0^2 \int_0^{2\pi} \left[zr^3 \right]_r^2 \, d\theta \, dr$$

$$= \int_0^2 \int_0^{2\pi} 2r^3 - r^4 \, d\theta \, dr = \int_0^2 \left[\theta(2r^3 - r^4) \right]_0^{2\pi} dr$$

$$= \int_0^2 4\pi r^3 - 2\pi r^4 \, dr = \left[\pi r^4 - \frac{2\pi}{5} r^5 \right]_0^2$$

$$= \pi \left(2^4 - \frac{2^6}{5} \right) = \frac{16\pi}{5}.$$

◆

Question 6.17. Let E be the region bounded by the planes $z = 1$, $z = 2$, and the hyperboloid $z^2 = \frac{1}{x^2+y^2}$ and evaluate $\iint_E x^2 + y^2 dV$.

ANSWER. We have the restrictions

$$z \in [1,2] \qquad\qquad 0 \le z^2 = \frac{1}{x^2 + y^2}$$

which in cylindrical gives

$$1 \le z \le 2 \qquad z^2 = \frac{1}{r^2} \implies r = \frac{1}{z} \qquad \theta \text{ unrestricted}$$

Thus $E_{\theta z r} = [0, 2\pi) \times [1,2] \times \left[0, \frac{1}{z}\right]$.

$$\iiint_E x^2 + y^2 \, dV$$

$$= \int_0^{2\pi} \int_1^2 \int_0^{\frac{1}{z}} r^2 \, r \, dr \, dz \, d\theta = \int_0^{2\pi} \int_1^2 \left[\frac{r^4}{4} \right]_0^{\frac{1}{z}} dr \, dz \, d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} \int_1^2 \frac{1}{z^4} \, dz \, d\theta = \frac{1}{4} \int_0^{2\pi} \left[-\frac{1}{3z^3} \right]_1^2 d\theta$$

$$= \frac{1}{4}\frac{1}{3} \int_0^{2\pi} -\frac{1}{8} + \frac{1}{1} \, d\theta = \frac{1}{4}\frac{1}{3}\frac{7}{8} \int_0^{2\pi} d\theta = \frac{1}{4}\frac{1}{3}\frac{7}{8} 2\pi = \frac{7}{48}\pi.$$

◆

6.1.2 Spherical Change of Variables

Spherical. The transformation T given by

$$T : \mathbb{R} \times [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$$
$$(r, \theta, \phi) \mapsto (r \sin\phi \cos\theta, \, r \sin\phi \sin\theta, \, r \cos\phi)$$

converts *from spherical to rectangular*.

Question 6.18. What is the Jacobian for the transformation T which maps from spherical to rectangular?

ANSWER.

$$\begin{aligned}
&\frac{\partial(r\sin\phi\cos\theta,\ r\sin\phi\sin\theta,\ r\cos\phi)}{\partial(r,\ \theta,\ \phi)} \\
&= \begin{vmatrix} \sin\phi\cos\theta & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\phi\sin\theta & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix} \\
&= \cdots = -r^2\sin\phi
\end{aligned}$$

Remember we take the absolute value of this Jacobian so the negative disappears. ◆

Proposition 6.19. Let T be the transformation from spherical to rectangular. Then

$$\begin{aligned}
&\iiint_R f(x,y,z)\ \mathrm{d}V \\
&= \iiint_{T^{-1}(R)} f(r\sin\phi\cos\theta,\ r\sin\phi\sin\theta,\ r\cos\phi)\ r^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.
\end{aligned}$$

Question 6.20. What is the *volume of the sphere* of radius R?

ANSWER. The sphere of radius R (in spherical) is $r = R$ and this equation encloses the region given by

$$E_{r\theta\phi} = [0,\ R] \times [0, 2\pi) \times [0, \pi].$$

Note we only have $0 \le r \le R$ so θ and ϕ are unconstrained. The volume of the sphere is thereby given by the integral

$$\begin{aligned}
\text{Volume}(E_{r\theta\phi}) &= \iiint_E 1\ \mathrm{d}V \\
&= \int_0^{\pi}\int_0^{2\pi}\int_0^{R} 1\ r^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{3}r^3\sin\phi\right]_0^R\ \mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{1}{3}R^3\int_0^{\pi}\int_0^{2\pi}\sin\phi\ \mathrm{d}\theta\,\mathrm{d}\phi = \frac{1}{3}R^3\int_0^{\pi}\left[\sin(\phi)\,\theta\right]_0^{2\pi}\ \mathrm{d}\phi \\
&= \frac{2}{3}\pi R^3\int_0^{\pi}\sin\phi\ \mathrm{d}\phi = \frac{2}{3}\pi R^3\left[-\cos\phi\right]_0^{\pi} = \frac{4}{3}\pi R^3.
\end{aligned}$$

◆

Question 6.21. Let B be the *unit ball* $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ and evaluate $\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$.

ANSWER. In spherical the unit ball is (easily) described as the region

$$E_{r\theta\phi} = [0, 1] \times [0, 2\pi) \times [0, \pi].$$

$$\begin{aligned}\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV &= \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{(r^2)^{\frac{3}{2}}}\, r^2 \sin\phi \, dr\, d\theta\, d\phi \\ &= \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{r^3} r^2 \sin\phi \, dr\, d\theta\, d\phi = \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^1 r^2 e^{r^3}\, dr \\ &= [-\cos\phi]_0^{\pi}\, [\theta]_0^{2\pi} \left[\frac{1}{3} e^{r^3}\right]_0^1 = \frac{4}{3}\pi(e - 1).\end{aligned}$$

◆

Question 6.22. What is the volume of the solid that lies *above* the cone $z = \sqrt{x^2 + y^2}$ and *below* the sphere $x^2 + y^2 + z^2 = z$.

ANSWER. To find the volume we integrate 1 inside the enclosed region.

Above the *cone* means $z \geq \sqrt{x^2 + y^2}$ which in *spherical* is

$$\begin{aligned} r\cos\phi &\geq \sqrt{r^2 \sin^2\phi \cos^2\theta + r^2 \sin^2\phi \sin^2\theta} \\ &\geq \sqrt{r^2 \sin^2\theta \left(\cos^2\theta + \sin^2\theta\right)} \\ &\geq r\sin\phi. \end{aligned}$$

Thereby $\cos\phi \geq \sin\phi$ and thus $\phi \in \left[0, \frac{\pi}{4}\right]$. The *sphere* $x^2 + y^2 + z^2 = z$ has writing in *spherical* $r^2 = r\cos\phi \implies r = \cos\phi$. Combining

$$\phi \in \left[0, \frac{\pi}{4}\right] \qquad r = \cos\phi \qquad \theta \text{ unrestricted}$$

gives $E_{\theta\phi r} = [0, 2\pi] \times \left[0, \frac{\pi}{4}\right] \times [0, \cos\phi]$.

$$\begin{aligned} &\iiint_E 1\, dV \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} 1\, r^2 \sin\phi \, dr\, d\phi\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} r^2 \sin\phi \, dr\, d\phi = 2\pi \int_0^{\frac{\pi}{4}} \sin\phi \left[\frac{r^3}{3}\right]_0^{\cos\phi} d\phi \\ &= \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \sin\phi \cos^3\phi \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{4}} \\ &= \frac{2\pi}{3} \frac{1}{4} \left[-\left(\frac{\sqrt{2}}{2}\right)^4 + 1\right] = \frac{\pi}{8}. \end{aligned}$$

◆

Question 6.23. Let E be the region *above* the xy-plane and *between* the spheres of radii 1 and 2 about the origin. Evaluate $\iiint_E \frac{1}{x^2+y^2+z^2}\,\mathrm{d}V$.

ANSWER. *Above* the xy-plane and *between* the spheres $r = 1$ and $r = 2$ corresponds to the (spherical) restrictions

$$\phi \in \left[0, \frac{\pi}{2}\right] \qquad r \in [1,2] \qquad \theta \text{ unrestricted.}$$

Thus $E_{\phi r \theta} = \left[0, \frac{\pi}{2}\right] \times [1,2] \times [0, 2\pi)$.

$$\iiint_E \frac{x^2+y^2+z^2}{}\mathrm{d}V$$

$$= \int_0^{2\pi} \int_1^2 \int_0^{\frac{\pi}{2}} \frac{1}{r^2}\, r^2 \sin\phi \,\mathrm{d}\phi \,\mathrm{d}r \,\mathrm{d}\theta = \int_0^{2\pi} \int_1^2 \int_0^{\frac{\pi}{2}} \sin\phi \,\mathrm{d}\phi \,\mathrm{d}r \,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_1^2 [-\cos\phi]_0^{\frac{\pi}{2}} \,\mathrm{d}r \,\mathrm{d}\theta = \int_0^{2\pi} \int_1^2 1 \,\mathrm{d}r \,\mathrm{d}\theta$$

$$= (2\pi - 0)(2-1) = 2\pi.$$

◆

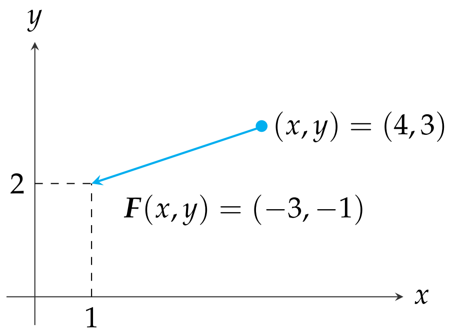
7 Vector Fields

A *vector field* is a function whose domain is from $\mathbb{R}^2$ or $\mathbb{R}^3$ and whose range is a set of vectors.

7.1 Vector Fields in 2D

Two Dimensional Vector Field. Let $D \subseteq \mathbb{R}^2$. A *vector field* on $\mathbb{R}^2$ is a function $\boldsymbol{F} : D \to \mathbb{R}^2$.

We visualize vector fields by drawing arrows representing the vectors $\boldsymbol{F}(x, y)$ starting at the point (x, y):



Although it is impossible to do this for *all* points (x, y) we can gain a reasonable impression of $\boldsymbol{F}$ by drawing *some* points.

Question 7.1. Let $\boldsymbol{F}(x, y) = \langle -y, x \rangle$ define a a vector field on $\mathbb{R}^2$. Draw $\boldsymbol{F}$.

ANSWER. First let us calculate some of the vectors at (arbitrary) points (Table 7.1) and plot the results (Figure 7.1).

◆

(x,y)	$\boldsymbol{F}(x,y)$	(x,y)	$\boldsymbol{F}(x,y)$	(x,y)	$\boldsymbol{F}(x,y)$
$(1,0)$	$\langle 0,1\rangle$	$(2,2)$	$\langle -2,2\rangle$	$(3,0)$	$\langle 0,3\rangle$
$(0,1)$	$\langle -1,0\rangle$	$(-2,2)$	$\langle -2,-2\rangle$	$(0,3)$	$\langle -3,0\rangle$
$(-1,0)$	$\langle 3,0\rangle$	$(-2,2)$	$\langle 2,-2\rangle$	$(-3,0)$	$\langle 0,-3\rangle$
$(0,-1)$	$\langle 1,0\rangle$	$(2,-2)$	$\langle 2,2\rangle$	$(0,-3)$	$\langle 3,0\rangle$

Table 7.1: For Question 7.1.

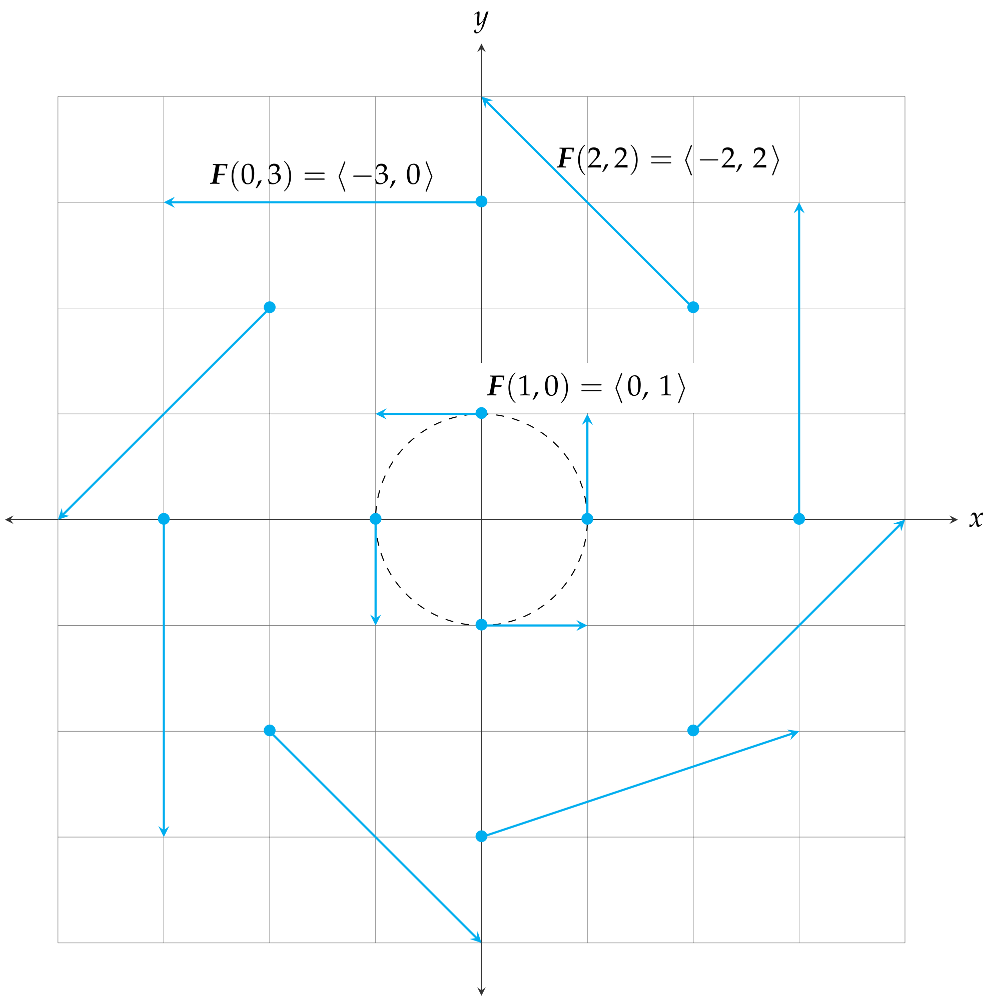


Figure 7.1: For Question 7.1.

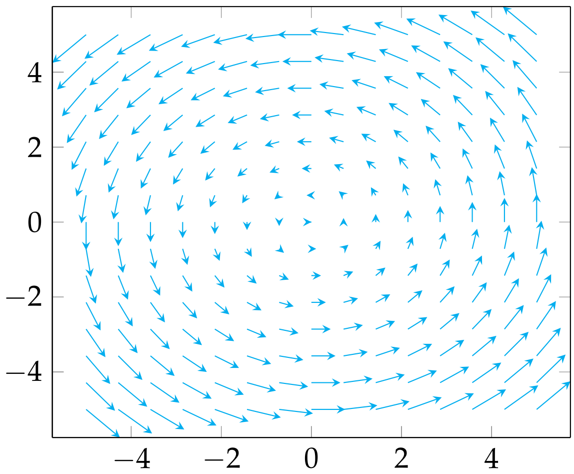


Figure 7.2: $\boldsymbol{F}(x, y) = \langle -y, x \rangle$.

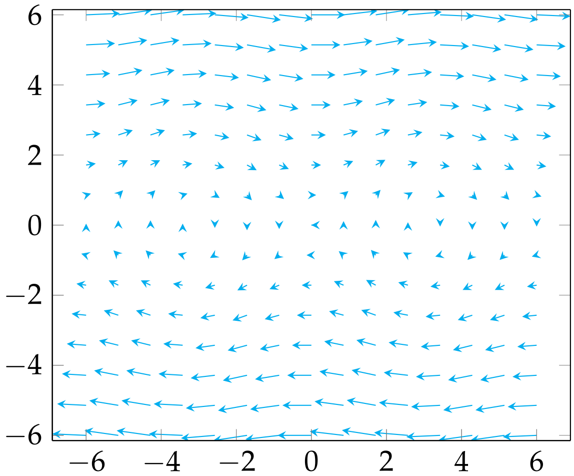


Figure 7.3: $\boldsymbol{F}(x, y) = \langle y, \sin x \rangle$.

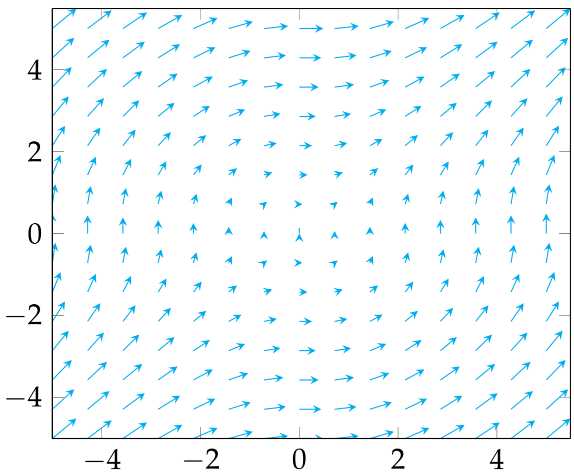
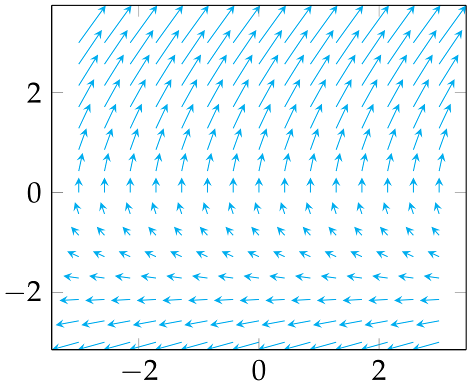


Figure 7.4: $\boldsymbol{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$.

Question 7.2. Match the plot with the vector field.[1]

1. $\boldsymbol{F}(x, y) = \langle x, -y \rangle$
2. $\boldsymbol{F}(x, y) = \langle y, x - y \rangle$
3. $\boldsymbol{F}(x, y) = \langle y, y + 2 \rangle$
4. $\boldsymbol{F}(x, y) = \langle \cos(x + y), x \rangle$



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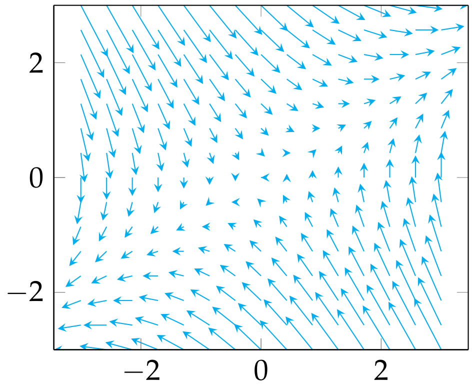

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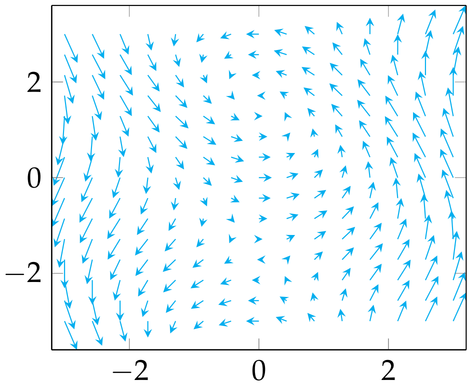

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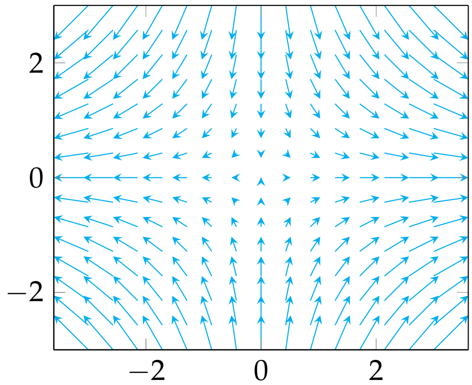


D

[1] Answer: 1D. 2B. 3A. 4C.

7.2 Vector Fields in 3D

Definition 7.3 (Three Dimensional Vector Field)**.** Let $D \subseteq \mathbb{R}^3$. A *vector field* on $\mathbb{R}^3$ is a function $\boldsymbol{F} : D \to \mathbb{R}^3$.

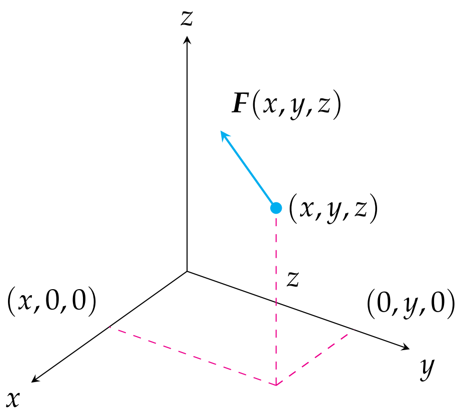


Figure 7.5: A single vector from this vector field.

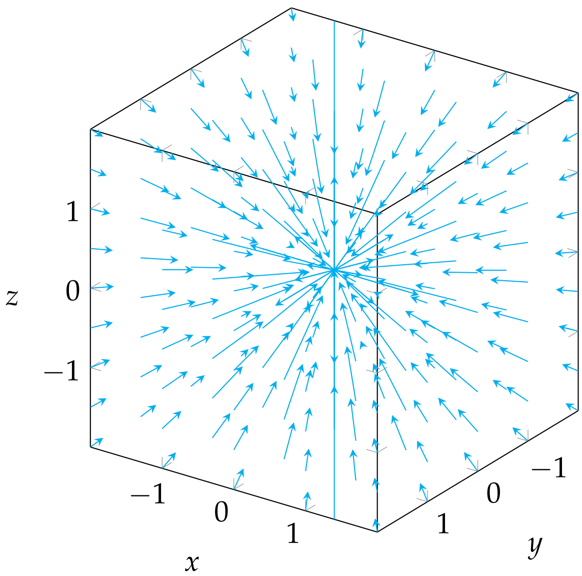


Figure 7.6: Gravitational Force Field:

$$\boldsymbol{F}(x, y, z) = \frac{-mMG}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \langle x,\, y,\, z \rangle\,.$$

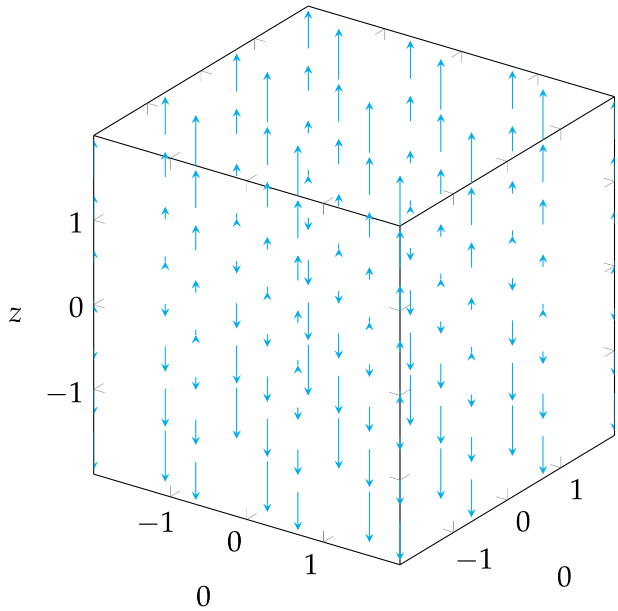


Figure 7.7: $\boldsymbol{F}(x, y, z) = \langle 0, 0, z \rangle$.

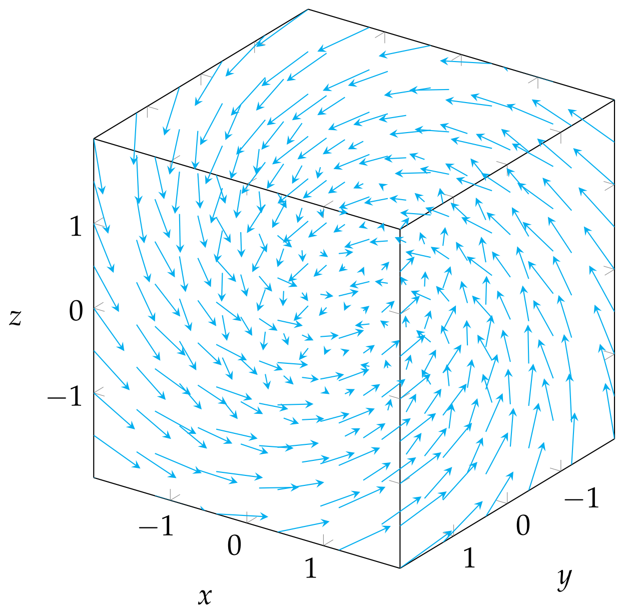


Figure 7.8: $\boldsymbol{F}(x, y, z) = \langle y, z, x \rangle$.

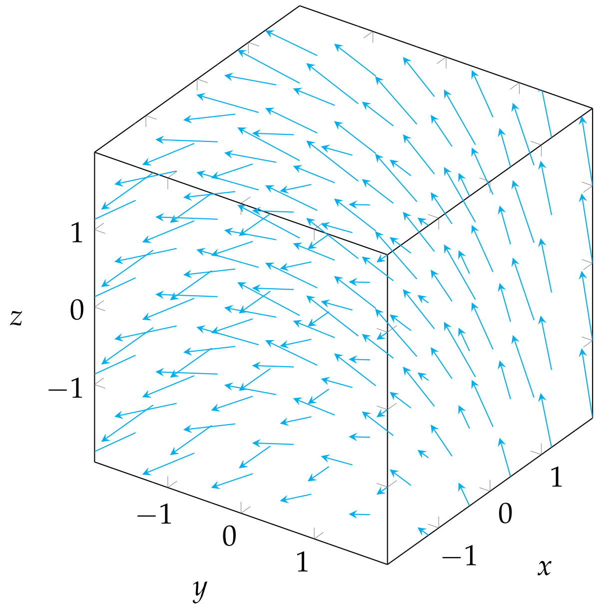


Figure 7.9: $\boldsymbol{F}(x, y, z) = \langle y,\ z,\ x \rangle$.

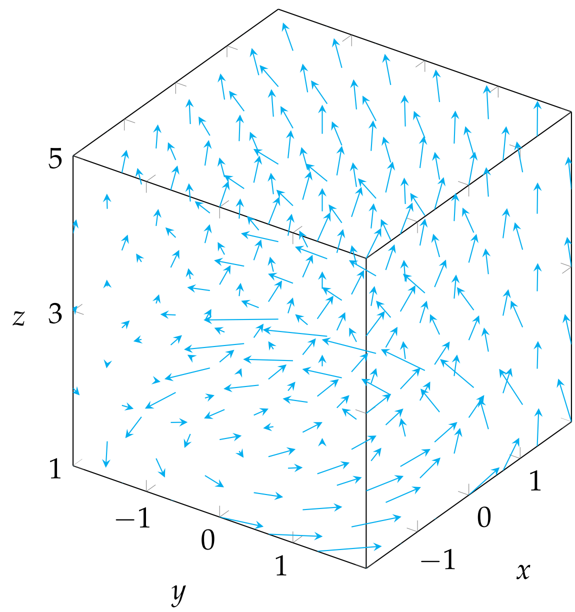
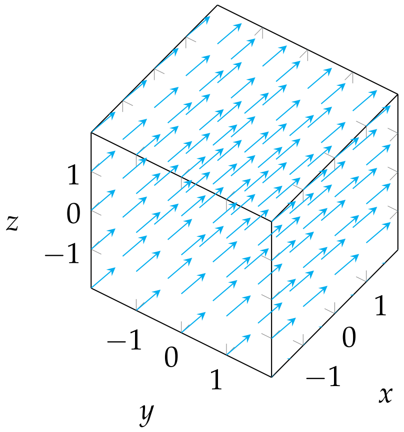


Figure 7.10: $\boldsymbol{F}(x, y, z) = \langle \frac{y}{z},\ -\frac{x}{z},\ \frac{z}{4} \rangle$.

Question 7.4. Match the plot with the gradient field.[2]

1. $\boldsymbol{F}(x, y) = \langle 1, 2, 3 \rangle$
2. $\boldsymbol{F}(x, y) = \langle x, y, 3 \rangle$
3. $\boldsymbol{F}(x, y) = \langle x, y, z \rangle$
4. $\boldsymbol{F}(x, y) = \langle 1, 2, z \rangle$



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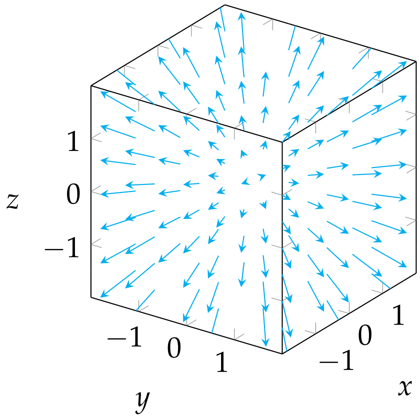

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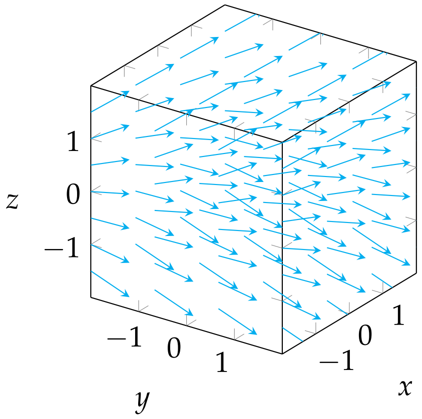

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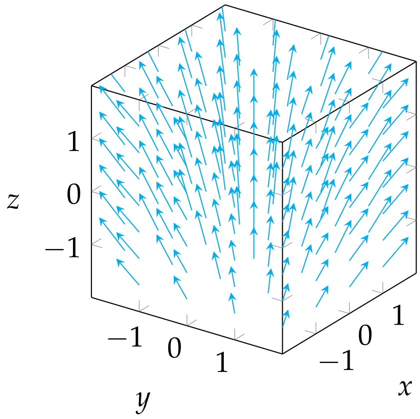


D

[2] Answer: 1A. 2D. 3B. 4C.

7.3 Gradient Field

Recall if f is a *scalar function* of two variables then the *gradient* of $f(x, y)$ is given by

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

and therefore ∇f defines a vector field on $\mathbb{R}^2$ — this is the so-called *gradient vector field*. Likewise $f(x, y, z)$ gives a gradient vector field in $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

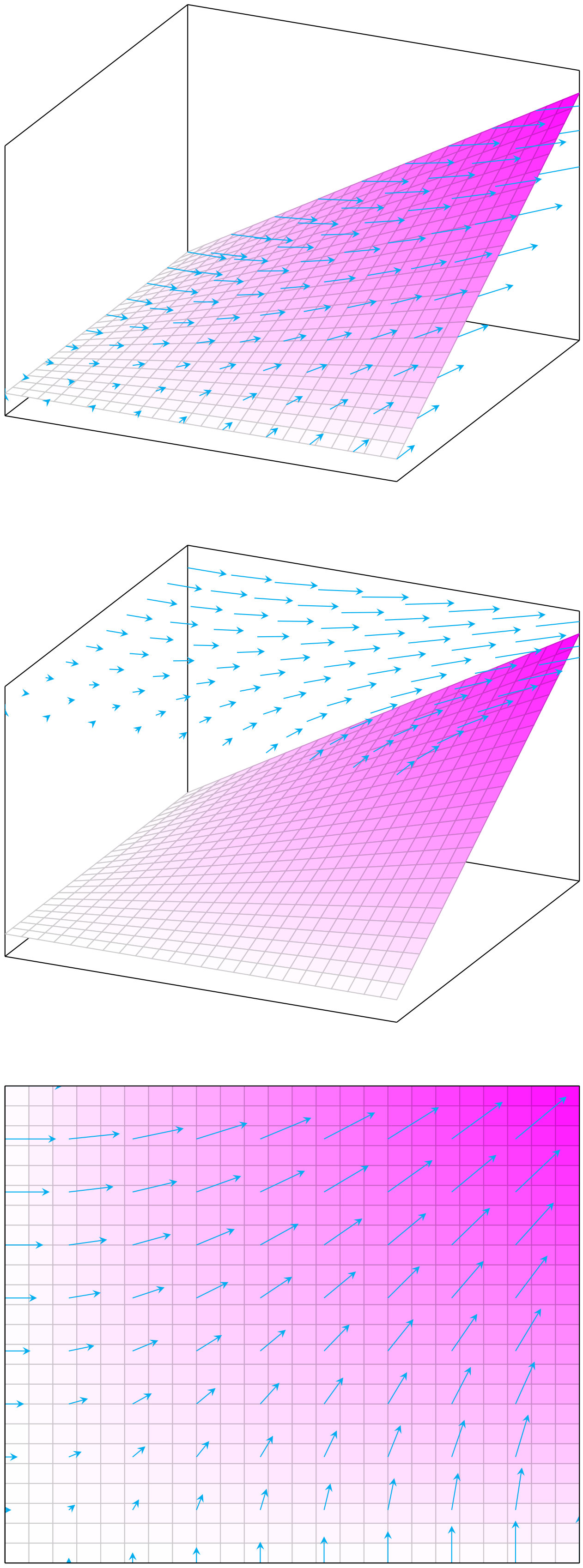


Figure 7.11: $f(x, y) = xy$ and its gradient field.

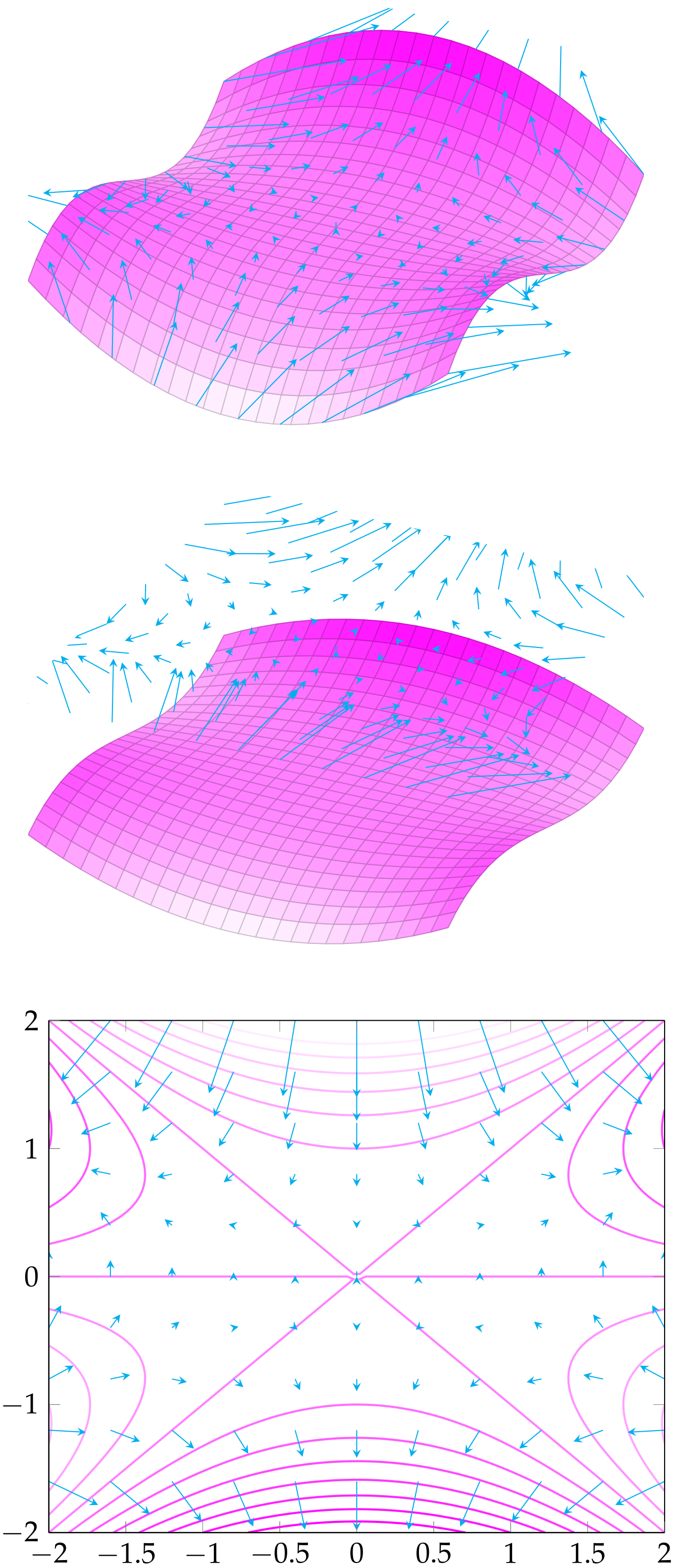


Figure 7.12: $x^2y - y^3$ and its gradient field $\langle 2x, x^2 - 3y^2 \rangle$.

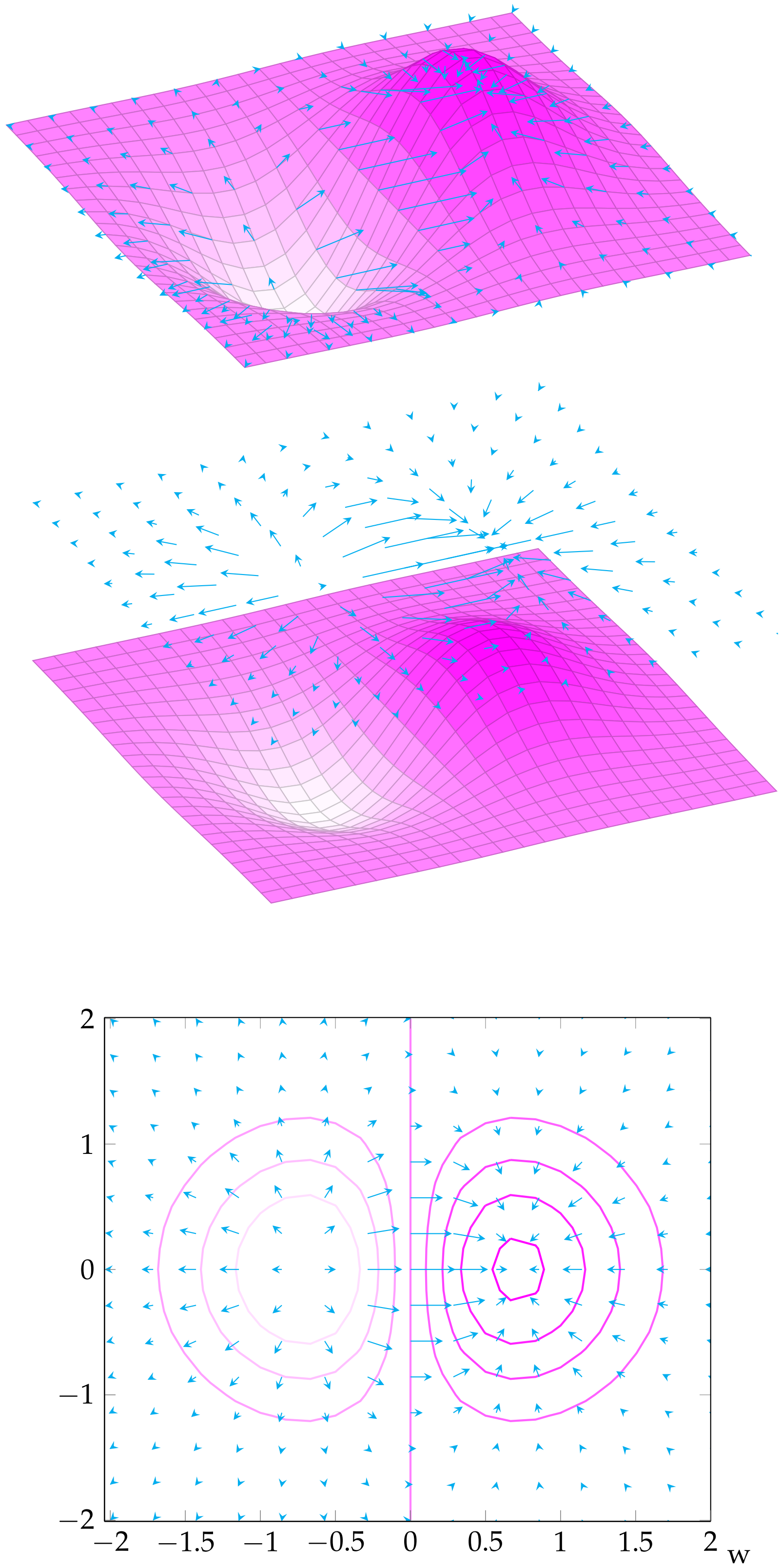
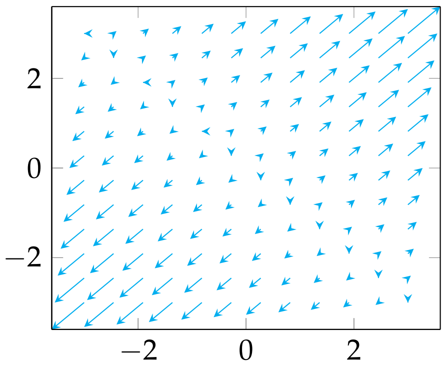


Figure 7.13: $xe^{-x^2-y^2}$ and its gradient field $\left\langle \frac{1-2x^2}{e^{x^2+y^2}}, \frac{-2xy}{e^{x^2+y^2}} \right\rangle$

Question 7.5. Match the plot with the gradient field.[3]

1. $\boldsymbol{F}(x, y) = x^2 + y^2$
2. $\boldsymbol{F}(x, y) = (x + y)^2$
3. $\boldsymbol{F}(x, y) = x(x + y)$
4. $\boldsymbol{F}(x, y) = \sin\sqrt{x^2 + y^2}$



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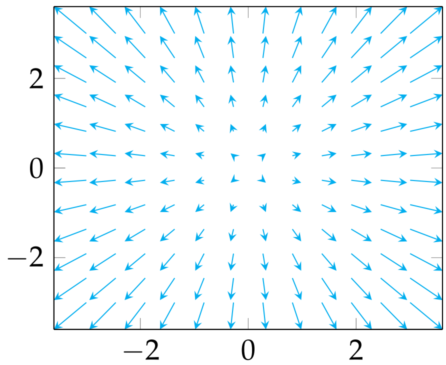

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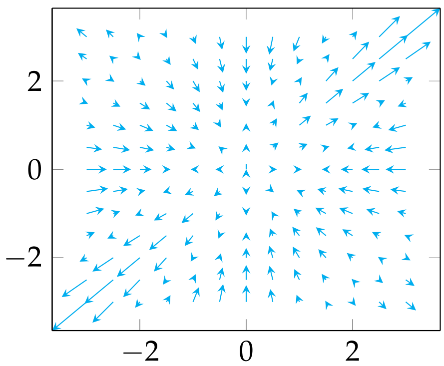

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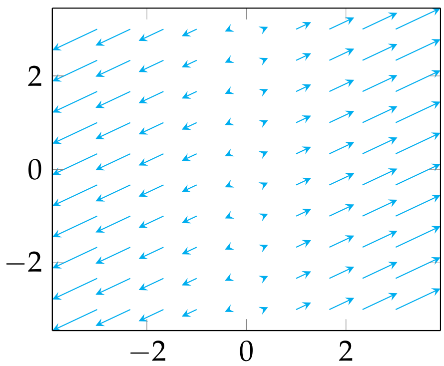


D

[3] Answer: 1B. 2A. 3D. 4C.

8 Line Integrals

8.1 Line Integrals

We have seen how to integrate over *regions* of various types. This lecture we study how to integrate over *curves*.

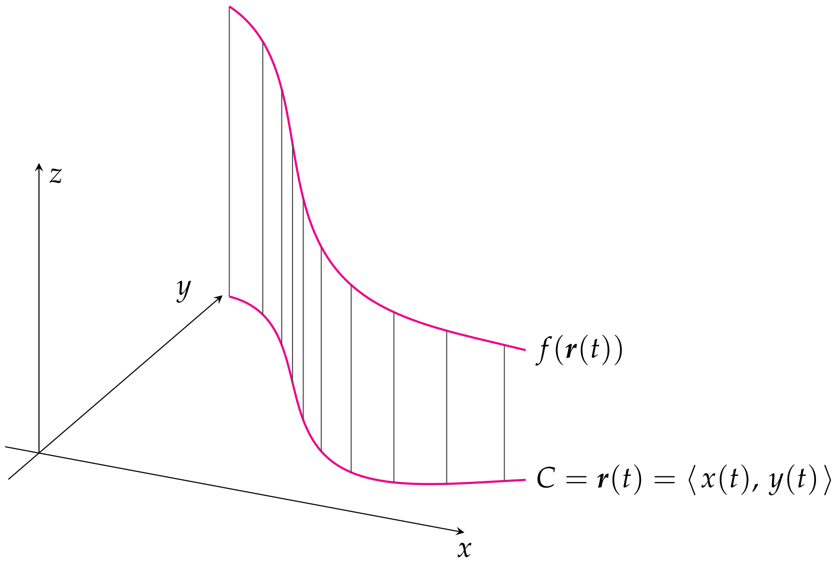


Figure 8.1: A *curtain* — the area under a curve in space.

Definition 8.1 (Line Integral)**.** If f is defined over a smooth curve C then the *line integral of f along C* is

$$\int_C f(x,y)\ \mathrm{d}s = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if the limit exists.

But how can we *practically* compute this limit? If we break $[a,b]$ into n intervals and let t_i^* be in the ith one then *the area under $f(\boldsymbol{r}(t))$ for $t \in [a,b]$*

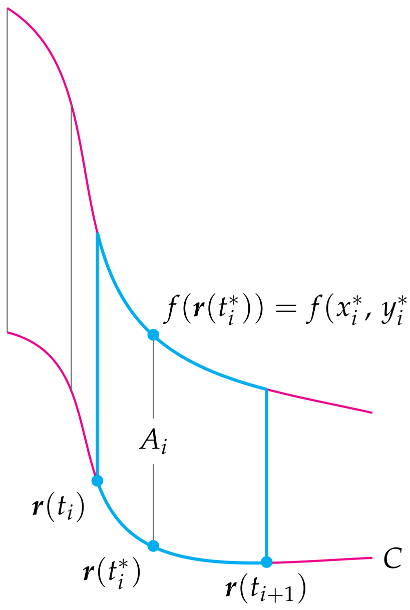


Figure 8.2: Let $\Delta s_i :=$ Distance from $\boldsymbol{r}(t_i)$ to $\boldsymbol{r}(t_{i+1})$ on C then

$$A_i \approx f(x_i^*, y_i^*)\, \Delta s_i.$$

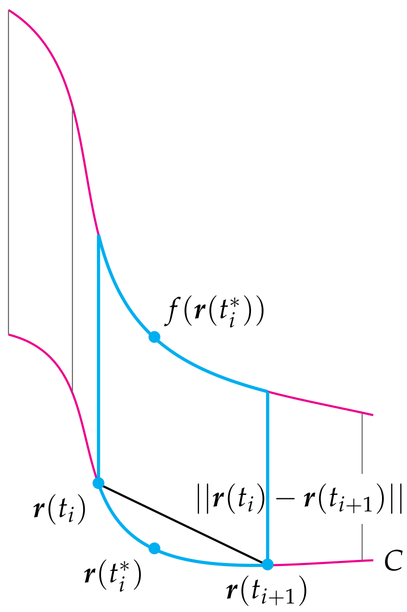


Figure 8.3: Area is *approximately* height times base or, when $t_i^* \in [t_i, t_{i-1}]$,

$$A_i \approx f(\boldsymbol{r}(t_i^*)) \cdot ||\boldsymbol{r}(t_i) - \boldsymbol{r}(t_{i+1})||.$$

is approximately

$$\begin{aligned}
&\sum_{i=1}^{n} f(\boldsymbol{r}(t^*)) \cdot ||\boldsymbol{r}(t_i) - \boldsymbol{r}(t_{i+1})|| \\
&\qquad = \sum_{i=1}^{n} f(\boldsymbol{r}(t^*)) \sqrt{(x(t_i) - x(t_{i+1}))^2 - (y(t_i) - y(t_{i+1}))^2} \\
&\qquad = \sum_{i=1}^{n} f(\boldsymbol{r}(t^*)) \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 - \left(\frac{\Delta y}{\Delta t}\right)^2} \Delta t.
\end{aligned}$$

As $\Delta t \to 0$ lines give successively better approximations of lengths on $\boldsymbol{r}$. To get $\Delta t \to 0$ we take $n \to \infty$ and so (borrowing arguments from last lecture where we determined the length of a curve)

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{n} f(\boldsymbol{r}(t^*)) \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 - \left(\frac{\Delta y}{\Delta t}\right)^2} \Delta t \\
&\qquad = \int_a^b f(\boldsymbol{r}(t)) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \,\mathrm{d}t = \int_a^b f(\boldsymbol{r}(t)) \cdot ||\boldsymbol{r}'(t)|| \,\mathrm{d}t.
\end{aligned}$$

Line Integral. If f is defined over a smooth curve $\boldsymbol{r}$ given by $\boldsymbol{r}(t) = \langle x(t), y(t) \rangle : t \in [a, b]$ then the *line integral of f along $\boldsymbol{r}$* is

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(\boldsymbol{r}(t)) \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \,\mathrm{d}t \tag{8.1}$$

$$= \int_a^b f(\boldsymbol{r}(t)) \cdot ||\boldsymbol{r}'(t)|| \,\mathrm{d}t. \tag{8.2}$$

Arc Length. If $\boldsymbol{r}$ is a parameterized smooth curve given by $\boldsymbol{r}(t) = \langle x(t), y(t) \rangle : t \in [a, b]$ then the *arc length of $\boldsymbol{r}$* is

$$\int_C \mathrm{d}s = \int_a^b ||\boldsymbol{r}'(t)|| \,\mathrm{d}t. \tag{8.3}$$

The value of the line integral does not depend on the parameterization of the curve provided that the curve is traversed *exactly once*.

Question 8.2. Let C be the upper half of the unit circle $x^2 + y^2 = 1$ and evaluate $\int_C (2 + x^2 y)\,\mathrm{d}s$.

ANSWER. A parametric form for C is $\boldsymbol{r}(t) = \langle \cos t, \sin t \rangle : t \in [0, \pi]$ and so $\boldsymbol{r}' = \langle -\sin t, \cos t \rangle$. Using (8.3)

$$\int_C (2 + x^2 y)\,\mathrm{d}s = \int_0^{\pi} (2 + \cos^2 t \sin t) \cdot ||\langle -\sin t, \cos t \rangle||\,\mathrm{d}t$$

$$= \int_0^{\pi} (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t}\ \mathrm{d}t$$

$$= \int_0^{\pi} (2 + \cos^2 t \sin t)\ \mathrm{d}t = \left[2t - \frac{\cos^3 t}{3} \right]_0^{\pi} = 2\pi + \frac{2}{3}.$$

◆

Proposition 8.3. Suppose C is the *piecewise-smooth curve* (like that of Figure 8.4) so that $C = C_1 \cup C_2 \cdots \cup C_n$ where the initial point of C_{i+1} is the terminal point of C_i. Then

$$\int_C f(x,y)\ \mathrm{d}s = \int_{C_1} f(x,y)\ \mathrm{d}s + \int_{C_2} f(x,y)\ \mathrm{d}s + \cdots + \int_{C_n} f(x,y)\ \mathrm{d}s.$$

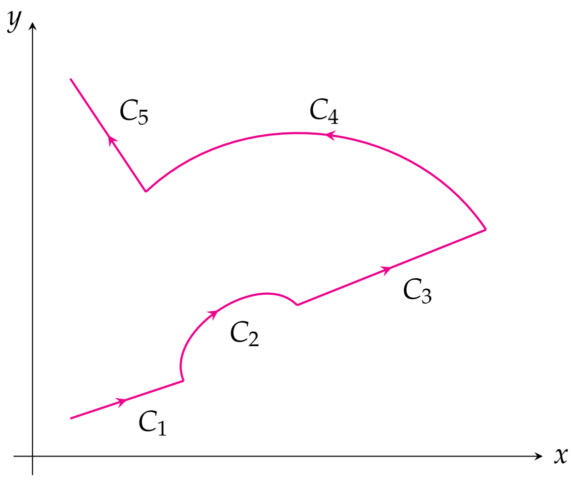


Figure 8.4: A piecewise-smooth curve.

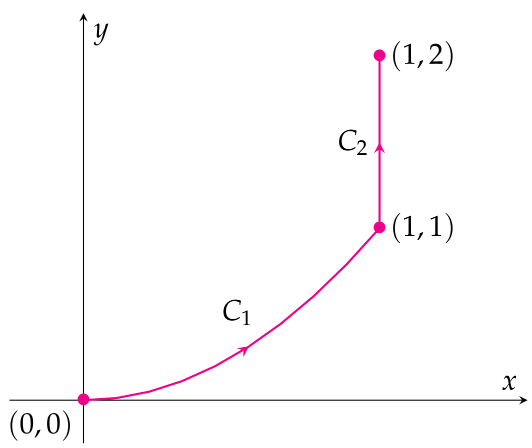


Figure 8.5: For Question 8.4.

Question 8.4. Evaluate $\int_C 2x\ \mathrm{d}s$ where C is the arc C_1 of $y = x^2$ from $(0,0)$

to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$. See Figure 8.5.

ANSWER. C_1 and C_2 are (easily) parameterized by

$$\boldsymbol{r}_1(t) = \left\langle t,\, t^2 \right\rangle : t \in [0,1] \qquad \boldsymbol{r}_2(t) = \langle 1,\, t\rangle : t \in [1,2].$$

Notice the endpoints match: $\boldsymbol{r}_1(1) = (1,1) = \boldsymbol{r}_2(1)$. By Proposition 8.3

$$\int_C f(x,y)\ \mathrm{d}s = \int_{C_1} f(x,y)\ \mathrm{d}s + \int_{C_2} f(x,y)\ \mathrm{d}s$$

so let us do the integrals separately.

Recall $\boldsymbol{r}_1 = \left\langle t,\, t^2 \right\rangle = \langle x(t),\, y(t)\rangle : t \in [0,1]$.

$$\int_{C_1} 2x\ \mathrm{d}s = \int_0^1 2x(t)\ ||\,\langle 1,\, 2t\rangle\,||\ \mathrm{d}t = \int_0^1 2t\,\sqrt{1^2+4t^2}\ \mathrm{d}t$$
$$= \left[\frac{1}{4}\,\frac{2}{3}\,(1+4t^2)^{\frac{3}{2}}\right]_0^1 = \frac{5\sqrt{5}-1}{6}$$

Recall $\boldsymbol{r}_2 = \langle 1,\, t\rangle = \langle x(t),\, y(t)\rangle : t \in [1,2]$ so

$$\int_{C_1} 2x\ \mathrm{d}s = \int_1^2 2\,x(t)\ ||\,\langle 0,\, 1\rangle\,||\ \mathrm{d}t = \int_1^2 2(1)\ \sqrt{0^2+1^2}\ \mathrm{d}t = \int_1^2 2\ \mathrm{d}t = 2.$$

Combining answers, we have $\int_C 2x\ \mathrm{d}s = \frac{5\sqrt{5}-1}{6} + 2.$ ◆

8.2 *Line Integrals with respect to x and y.*

Suppose we prefer to break x and y into intervals rather than t. These are called *line integrals of f along C with respect to x and y*:

$$\int_C f(x,y)\,\mathrm{d}x = \lim_{n\to\infty}\sum_i^n f(x_i^*,\, y_i^*)\Delta x_i \qquad \int_C f(x,y)\,\mathrm{d}y = \lim_{n\to\infty}\sum_i^n f(x_i^*,\, y_i^*)\Delta y_i$$

Proposition 8.5. If $C = \boldsymbol{r}(t) : t \in [a,b]$ then

$$\int_C f(x,y)\ \mathrm{d}x = \int_a^b f(x(t),\, y(t))\ \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$$

$$\int_C f(x,y)\ \mathrm{d}x = \int_a^b f(x(t),\, y(t))\ \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t$$

Because it frequently happens that these type of integrals occur together it is customary to write

$$\int_C P(x,y)\mathrm{d}x + \int_C Q(x,y)\,\mathrm{d}y = \int_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y.$$

Question 8.6. Find $\int_{C_1} y^2\,\mathrm{d}x + x\,\mathrm{d}y$ where

$$C_1 = \{\text{Line segment from } (-5,-3) \text{ to } (0,2)\}.$$

ANSWER. The line segment is parameterized by

$$r_1(t) = \langle 5t-5,\, 5t-3\rangle = \langle x(t),\, y(t)\rangle : t \in [0,1]$$

and so $x = 5t-5 \implies \frac{\mathrm{d}x}{\mathrm{d}t} = 5 \implies \mathrm{d}x = 5\mathrm{d}t$, $y = 5t-3 \implies \frac{\mathrm{d}y}{\mathrm{d}t} = 5 \implies \mathrm{d}y = 5\mathrm{d}t$.

$$\begin{aligned}&\int_{C_1} y^2\,\mathrm{d}x + x\,\mathrm{d}y\\ &\quad= \int_0^1 (5t-5)^2(5\mathrm{d}t) + (5t-5)(5\mathrm{d}t) = 5\int_0^1 (25t^2-25t+4)\,\mathrm{d}t\\ &\quad= 5\left[\frac{25}{3}t^3 - \frac{25}{2}t^2 + 4t\right]_0^1 = -\frac{5}{6}.\end{aligned}$$

◆

Question 8.7. Evaluate $\int_{C_2} y^2\,\mathrm{d}x + x\,\mathrm{d}y$ where

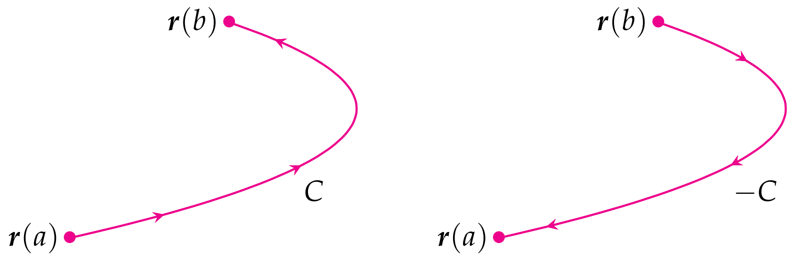
$$C_2 = \left\{\text{Arc of parabola } x = 4-y^2 \text{ from } (-5,-3) \text{ to } (0,2)\right\}.$$

ANSWER. We can parametrize C_2 by $C_2(t) = \left\langle 4-t^2,\, t\right\rangle : t \in [-3,2]$ and so $x = 4-t^2 \implies \frac{\mathrm{d}x}{\mathrm{d}t} = -2t \implies \mathrm{d}x = -2\mathrm{d}t$, and $y = t \implies \frac{\mathrm{d}y}{\mathrm{d}t} = 1 \implies \mathrm{d}y = \mathrm{d}t$. Finally

$$\begin{aligned}&\int_C y^2\,\mathrm{d}x + x\,\mathrm{d}y\\ &\quad= \int_{-3}^2 y^2(-2y)\,\mathrm{d}y + (4-y^2)\,\mathrm{d}y = \int_{-3}^2 (-2y^3 - y^2 + 4)\,\mathrm{d}y\\ &\quad= \left[-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right]_{-2}^2 = 40 + \frac{5}{6}.\end{aligned}$$

◆

Notice different answers are obtained even though C_1 and C_2 have the same endpoints. In general the value of a line integral depends not just on the endpoints, but also on the path itself.



Moreover, reversing direction (or more generally orientation) changes the parity of the integral:

$$\int_C f(x,y)\,\mathrm{d}s = -\int_{-C} f(x,y)\,\mathrm{d}s.$$

Exercise 8.1. Confirm this for the previous question by reversing the direction of the paths.

8.3 *Line Integrals in Space*

Proposition 8.8. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a surface and C is a curve (in space) parameterized by $\langle x(t),\, y(t),\, z(t)\rangle : t \in [a,b]$ then

$$\int_C f(x,y,z)\,\mathrm{d}s$$
$$= \int_a^b f(x(t),\, y(t),\, z(t))\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

If we let $\boldsymbol{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$ then we can rewrite this integral as

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(\boldsymbol{r}(t))\,||\boldsymbol{r}'(t)||\,\mathrm{d}t.$$

Lemma 8.9. If C is a curve parameterized by $\boldsymbol{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$ then

$$\int_C 1\,\mathrm{d}s = \int_a^b |\boldsymbol{r}'(t)|\,\mathrm{d}t = \text{Length}(C).$$

Question 8.10. Evaluate $\int_C y\sin z\,\mathrm{d}s$ where C is the circular helix given by the equations $\boldsymbol{r}(t) = \langle \cos t,\, \sin t,\, t\rangle : t \in [02\pi]$.

ANSWER.

$$\int_C y\sin z\,\mathrm{d}s$$

$$= \int_0^{2\pi} (\sin t) \sin t \cdot ||\langle -\sin t, \cos t, 1\rangle|| \, dt$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} \, dt = \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t) \, dt$$

$$= \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t) \, dt = \frac{\sqrt{2}}{2} \left[t - \frac{1}{2} \sin 2t \right]_0^{2\pi} = \sqrt{2}\pi.$$

◆

Proposition 8.11. A parameterization of the line segment connecting P and $Q \in \mathbb{R}^3$ is given by $\boldsymbol{r}(t) = (1-t)P + tQ$.

PROOF. Start at P and travel in the direction $Q - P$ towards Q. Regarding $t \in [0,1]$ as the "percentage travelled along $Q - P$" then the parameterization can be written $\boldsymbol{r}(t) = P + t(Q - P) = (1-t)P + tQ$. ■

Question 8.12. $\int_C y dx + z dy + x dz$ where $C = C_1 \cup C_2$ and

$$C_1 = \{\text{line segment from } (2,0,0) \text{ to } (3,4,5)\},$$

$$C_2 = \{\text{line segment from } (3,4,5) \text{ to } (3,4,0)\}.$$

ANSWER. We can parameterize C_1 by

$$\boldsymbol{r}_1(t) = (1-t)\langle 2, 0, 0\rangle + t\langle 3, 4, 5\rangle = \langle 2+t, 4t, 5t\rangle : t \in [0,1]$$

where $\frac{dx}{dt} = 1 \implies dx{=}dt$, $\frac{dy}{dt} = 4 \implies dy = 4dt$, $\frac{dz}{dt} = 5 \implies dz = 5\,dt$.

$$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 (4t)\,dt + (5t)\,4dt + (2+t)\,5dt$$

$$= \int_0^1 (10 + 29t)\,dt = \left[10t + \frac{29}{2}t^2\right]_0^1 = \frac{49}{2}.$$

We can parameterize C_2 by

$$\boldsymbol{r}_2(t) = (1-t)\langle 3, 4, 5\rangle + t\langle 3, 4, 0\rangle = \langle 3, 4, 5-5t\rangle : t \in [0,1]$$

and so $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, and $\frac{dz}{dt} = -5$. Thus

$$\int_{C_2} y\,dx + z\,dy + x\,dz = \int 3(-5)\,dt = -15.$$

Adding these values gives $\int_C y\,dx + z\,dy + x\,dz = \frac{19}{2}$. ◆

8.4 Line Integrals of Vector Fields

Consider the work done by a *force field* F by moving a particle along a path $\boldsymbol{r}(t)$. For instance, the work done by water moving rubber duck along a path. (See Figure 8.7.) Recall the work done by a *force* $\boldsymbol{F}$ to displace (i.e. move in a straight line) a particle from P to Q in $\mathbb{R}^3$ is $W = \boldsymbol{F} \cdot \boldsymbol{D}$ where $\boldsymbol{D} := P - Q$ is the *displacement vector*.

This means the work done by $\boldsymbol{F}$ moving a particle from $\boldsymbol{r}(t_i^*)$ to $\boldsymbol{r}(t_{i+1}^*)$ *along* C is approximately $\boldsymbol{F}(\boldsymbol{r}(t_i^*)) \cdot \Delta s_i \hat{\boldsymbol{r}}'(t_i^*)$ as $\Delta s_i \hat{\boldsymbol{r}}'(t_i^*) \approx \boldsymbol{r}(t_{i+1}^*)$. (This is a linear approximation.) The *total work* to move the particle from $\boldsymbol{r}(a)$ to $\boldsymbol{r}(b)$ along $\boldsymbol{r}(t)$ is approximately

$$\sum_{i=1}^{n} \boldsymbol{F}(\boldsymbol{r}(t_i^*)) \cdot [\Delta s_i \hat{\boldsymbol{r}}(t_i^*)]$$

and if we say $\Delta s_1 = \Delta s_2 = \cdots \Delta = s_n$ (remember, we are choosing the intervals in the first place) then we simplify further to

$$\sum_{i=1}^{n} \boldsymbol{F}(\boldsymbol{r}(t_i^*)) \cdot \hat{\boldsymbol{r}}(t_i^*)\, \Delta s.$$

(Here "$\cdot$" is the dot product.) Taking $\Delta s \to 0$ or equivalently $n \to \infty$ gives

$$\lim_{n\to\infty} \sum_{i=1}^{n} \boldsymbol{F}(\boldsymbol{r}(t_i^*)) \cdot \hat{\boldsymbol{r}}(t_i^*)\, \Delta s = \int_C \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \hat{\boldsymbol{r}}'(t)\, \mathrm{d}s$$

Recall from Proposition 8.8 that

$$\int_C f(x,y,z)\, \mathrm{d}s = \int_a^b f(\boldsymbol{r}(t))\, ||\boldsymbol{r}'(t)||\, \mathrm{d}t.$$

This means, setting $f(x,y,z) = \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \hat{\boldsymbol{r}}'(t)$, that

$$\int_C \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \hat{\boldsymbol{r}}'(t)\, \mathrm{d}s = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \hat{\boldsymbol{r}}'(t)\, ||\boldsymbol{r}'(t)||\, \mathrm{d}t$$

$$= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{\boldsymbol{r}'(t)}{||\boldsymbol{r}'(t)||}\, ||\boldsymbol{r}'(t)||\, \mathrm{d} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, \mathrm{d}t.$$

Line Integral through vector field. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ define a smooth curve C given by $\boldsymbol{r}(t) : t \in [a,b]$. Then the *line integral of* $\boldsymbol{F}$ *along* C is

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, \mathrm{d}t$$

where "$\cdot$" is the *dot product*.

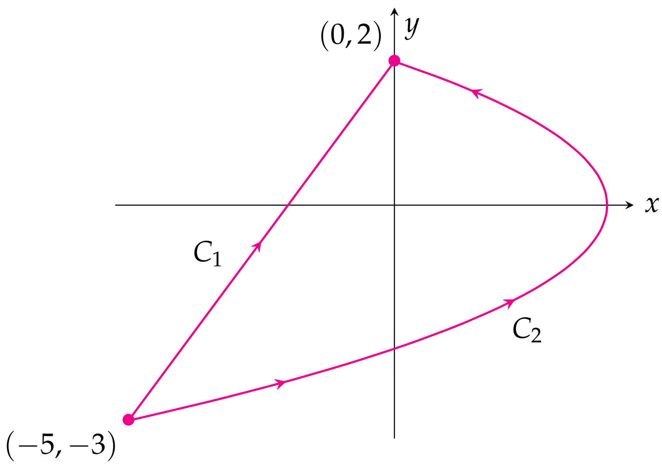


Figure 8.6: For Question 8.6 and 8.7.

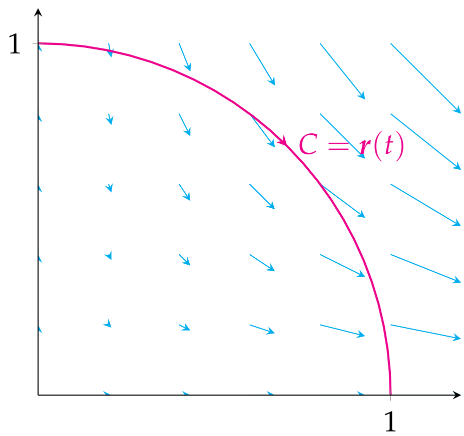


Figure 8.7: A particle moving along a path in a vector field.

Question 8.13. Find the work done by $\boldsymbol{F}(x,y) = \left\langle x^2, -xy \right\rangle$ in moving a particle along $\boldsymbol{r}(t) = \langle \cos t, \sin t \rangle : t \in [0, \pi/2]$.

ANSWER.

$$\boldsymbol{F}(\boldsymbol{r}(t)) = \left\langle \cos^2 t, -\cos t \sin t \right\rangle, \quad \boldsymbol{r}'(t) = \langle -\sin t, \cos t \rangle : t \in [0, \pi/2].$$

Therefore the work done is

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$
$$= \int_0^{\frac{\pi}{2}} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\,\mathrm{d}t = \int_0^{\frac{\pi}{2}} (-2\cos^2 t \, \sin t)\,\mathrm{d}t = 2\left[\frac{\cos^3 t}{3}\right]_0^{\frac{\pi}{2}} = -\frac{2}{3}.$$

◆

Question 8.14. Find $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ when C is the curve $\boldsymbol{r}(t) = \left\langle t, t^2, t^3 \right\rangle : t \in [0, 1]$.

ANSWER. We have

$$\boldsymbol{r}(t) = \left\langle t, t^2, t^3 \right\rangle, \qquad \boldsymbol{r}'(t) = \left\langle 1, 2, 3t^2 \right\rangle, \qquad \boldsymbol{F}(\boldsymbol{r}(t)) = \left\langle t^3, t^5, t^4 \right\rangle.$$

And so

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\,\mathrm{d}t = \int_0^1 (t^3 + 5t^6)\,\mathrm{d}t = \left[\frac{1}{4}t^4 + \frac{5}{7}t^7\right]_0^1 = \frac{27}{28}.$$

◆

9 Green's Theorem

Green's Theorem gives the relationship between a line integral around a *simple*, *closed*, *orientable* curve C and a double integral over the planar region D *bounded by* C. Green's Theorem should be regarded as the counterpart of the *Fundamental Theorem of Calculus* for double integrals.

Green's Theorem. Let C be *positively oriented*, *piecewise-smooth*, *simple*, *closed* curve in $\mathbb{R}^2$ and let D be the region bounded by C. If P and Q have continuous partial derivates on an open region that contains D, then

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,\mathrm{d}A.$$

(Recall that $\boldsymbol{F} = \langle P, Q\rangle$ so the left-hand-side is also equal to $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$.)

Notation. When C is *positively oriented* and *closed* we write

$$\int_C \boldsymbol{F}\cdot\langle \mathrm{d}x, \mathrm{d}y\rangle = \oint_C \boldsymbol{F}\cdot\langle \mathrm{d}x, \mathrm{d}y\rangle\,.$$

Notation. When D is bounded by the *positively oriented* closed curve C then we say

$$\partial D = C.$$

That is ∂D is read "the boundary of D."

9.1 Proof of Green's Theorem

We will prove Green's "for fun" (i.e. not testable) — many examples will come after. Green's Theorem is *hard* to prove *in general*, but we can give

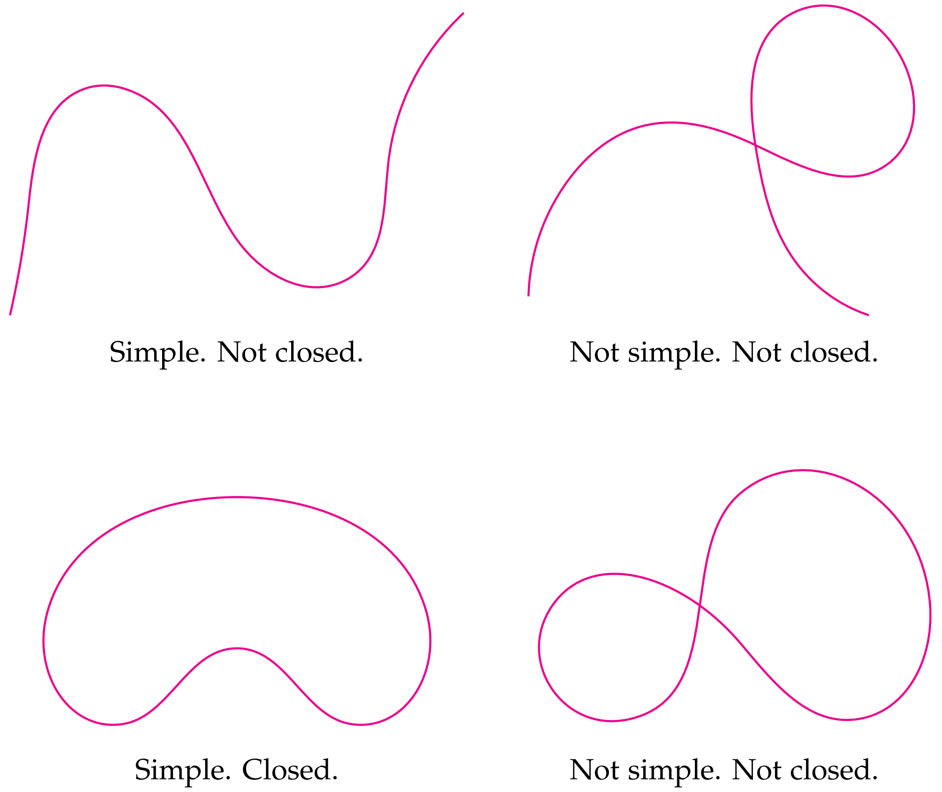


Figure 9.1: The various kinds of curves.

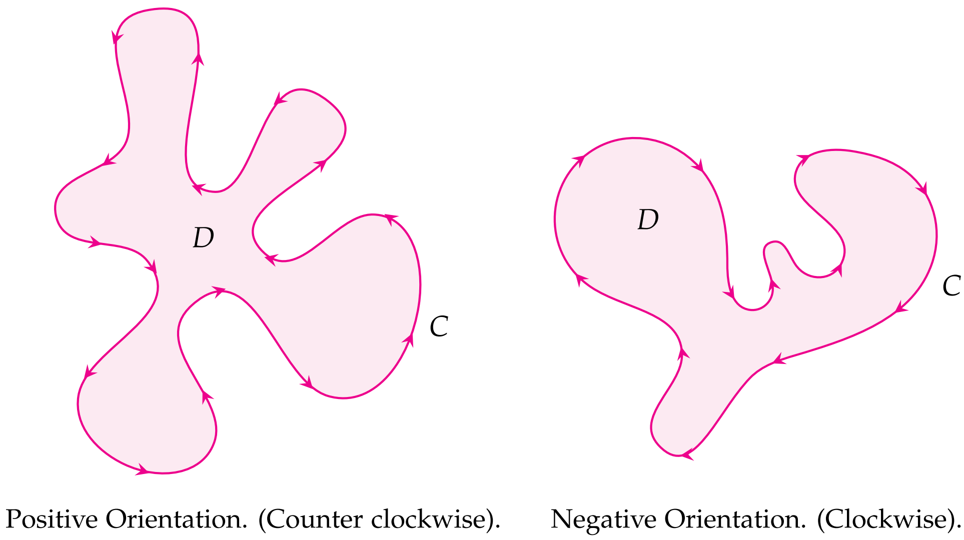


Figure 9.2: C is the *boundary* and D is the region given by the interior (including boundary).

a proof for the special case where the region C is *both* Type I and Type II. Such a region is called a *Simple Region*.

Simple Region. A region that is both Type I and Type II is called a *simple region*.

To prove Green's Theorem it suffices to show $\oint_C P\,\mathrm{d}x = -\iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A$ and $\oint_C Q\,\mathrm{d}y = \iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A$.

Lemma 9.1. $\oint_C P\,\mathrm{d}x = -\iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A$.

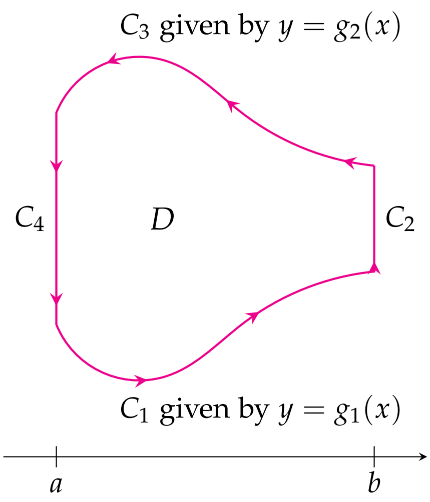


Figure 9.3: For Lemma 9.1

PROOF. *Assume that* D is a Type I region given by $D = [a, b] \times [g_1(x), g_2(x)]$. Thereby

$$\iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\,\mathrm{d}x.$$

We can break up ∂D into four pieces as in Figure 9.3. Since $C = C_1 \cup C_2 \cup C_3 \cup C_4$ we have

$$\oint_C P\,\mathrm{d}x$$

$$= \oint_{C_1} P(x, y)\,\mathrm{d}x + \oint_{C_2} P(x, y)\,\mathrm{d}x + \oint_{C_3} P(x, y)\,\mathrm{d}x + \oint_{C_4} P(x, y)\,\mathrm{d}x$$

Notice $\mathrm{d}x = 0$ for C_2 and C_4

$$= \oint_a^b P(x, g_1(x))\,\mathrm{d}x + 0 + \oint_b^a P(x, g_2(x))\,\mathrm{d}x + 0$$

$$= \oint_a^b P(x, g_1(x))\,\mathrm{d}x - \oint_a^b P(x, g_2(x))\,\mathrm{d}x$$

$$= \oint_a^b [P(x, g_2(x)) - P(x, g_1(x))]\, dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\, dy\, dx = \iint_D \frac{\partial P}{\partial y}\, dA$$

An analogous argument demonstrates

Lemma 9.2. $\oint_C Q\, dx = \iint_D \frac{\partial Q}{\partial x}\, dA$.

Green's Theorem follows. ■

Question 9.3. Let C be the triangular region comprised of the line segments $(0,0)$ to $(1,0)$, $(1,0)$ to $(0,1)$, and $(0,1)$ to $(0,0)$ — Figure 9.4. Evaluate $\oint_C x^4\, dx + xy\, dy$. (Note we could also evaluate this integral via three line integrals.)

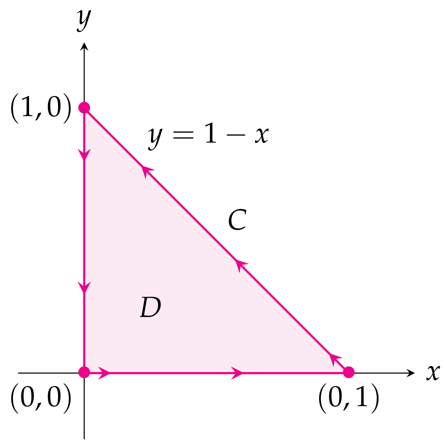


Figure 9.4: For Question 9.3

ANSWER. C's orientation is counter-clockwise and thus *positively oriented* as we require it to be. We have $\boldsymbol{F} = \langle x^4, xy \rangle = \langle P, Q \rangle$ therefore $\frac{\partial Q}{\partial x} = y$ and $\frac{\partial P}{\partial x} = 4x^3$

$$\oint_C x^4\, dx + xy\, dy$$

$$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_0^1 \int_0^{1-x} (y - 0)\, dy\, dx = \int_0^1 \left[\frac{1}{2}y^2\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\, dx = \left[-\frac{1}{6}(1-x)^3\right]_0^1 = \frac{1}{6}.$$

◆

Question 9.4. $\oint_C \langle 3y - e^{\sin x}, 7x + \sqrt{y^4 + 1} \rangle \cdot \langle dx, dy \rangle$ where C is the circle $x^2 + y^2 = 9$.

ANSWER. C can be parameterized by $C(t) = \langle 2\cos t, 2\sin t\rangle$ and indeed this is a positively oriented, closed curve.

$$\begin{aligned}
&\oint_C \langle 3y - e^{\sin x}, 7x + \sqrt{y^4+1}\rangle \cdot \langle dx, dy\rangle \\
&= \iint_D \frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin x})\, dA \\
&= \int_0^{2\pi}\int_0^3 (7-3)\, r\, dr d\theta = 4\int_0^{2\pi} d\theta \int_0^3 r\, dr \\
&= 4(2\pi - 0)\left(\frac{3^2}{2} - 0\right) = 36\pi.
\end{aligned}$$

◆

Proposition 9.5. If $P(x,y) = Q(x,y) = 0$ on C. That is, if $P(C(t)) = Q(C(t)) = 0$ for $t \in \operatorname{dom}(C)$ then

$$\iint_D \frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\, dA = \oint_C \langle P, Q\rangle \cdot \langle dx, dy\rangle = 0.$$

Proposition 9.6. Let D be a region so that ∂D is closed and simple.

$$\text{Area}(D) = \oint_{\partial D} x\, dy = -\oint_{\partial D} y\, dx = \frac{1}{2}\oint_{\partial D} x\, dy - y\, dx.$$

PROOF. Recall the area of D is given by $\iint_D 1\, dA$. Thus we if we choose

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \implies \iint_D 1 dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_{\partial D} P\, dx + Q\, dy.$$

For each of the following

$$P(x,y) = 0 \qquad P(x,y) = -y \qquad P(x,y) = -\frac{1}{2}y$$

$$Q(x,y) = x \qquad Q(x,y) = 0 \qquad Q(x,y) = \frac{1}{2}x$$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ is satisfied. The result follows. ■

Question 9.7. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

ANSWER. The ellipse is parameterized by $C(t) = \langle x, y\rangle = \langle a\cos t, b\sin t\rangle$ for $t \in [0, 2\pi]$. (Verify that C is closed and positively oriented.) Using Proposition 9.6 we have

Area(ellipse)

$$= \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{1}{2}\int_0^{2\pi}(a\cos t)(b\cos t)\,\mathrm{d}t - (b\sin t)(-a\sin t)\,\mathrm{d}t$$
$$= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t = \frac{ab}{2}\int_0^{2\pi}\mathrm{d}t = \pi ab.$$

◆

9.2 Green's Theorem for finite unions of simple regions

Although we have proved Green's theorem for the case that D is a simple region, we can extend it to the case where D is a *finite union* of simple regions like that of Figure 9.5. Applying Green's Theorem to D_1 and D_2 separately we get

$$\int_{C_1 \cup C_3} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A,$$

$$\int_{C_2 \cup -C_3} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A.$$

Since the line integrals along C_3 and $-C_3$ cancel we have

$$\int_{C_1 \cup C_2} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A$$

which is Green's Theorem for $D = D_1 \cup D_2$ with boundary $C = C_1 \cup C_2$. Moreover, the same argument can be modified to establish Green's Theorem for finite unions of non-overlapping simple regions like that Figure 9.6.

Question 9.8. Evaluate $\oint_C y^2\,\mathrm{d}x + 3xy\,\mathrm{d}y$ where C is the semiannular *simply connected* region D in the *upper-half-plane* between circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

ANSWER. D itself may not be simple but the y-axis divides it into two simple regions none-the-less. This means we can proceed as usually. In polar $D_{r\theta} = [1, 2] \times [0, \pi]$ and therefore Green's gives

$$\oint_C y^2\,\mathrm{d}x + 3xy\,\mathrm{d}y = \iint_D \frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\,\mathrm{d}A$$
$$= \iint_D y\,\mathrm{d}A = \int_0^{\pi}\int_1^2 (r\sin\theta) r\,\mathrm{d}r\mathrm{d}\theta$$
$$= \int_0^{\pi}\sin\theta\,\mathrm{d}\theta \int_1^2 r^2\,\mathrm{d}r = [-\cos\theta]_0^{\pi}\left[\frac{1}{3}r^3\right]_1^2 = \frac{14}{3}.$$

◆

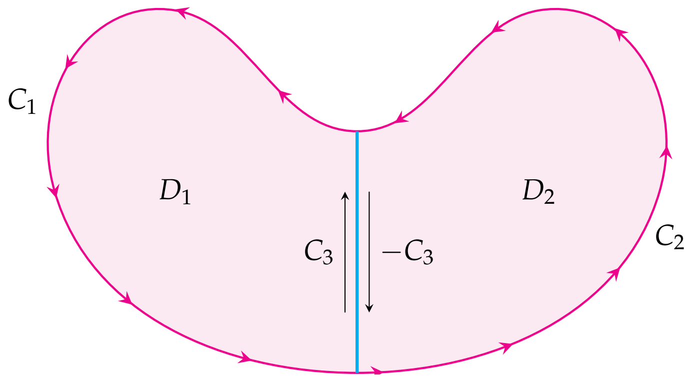


Figure 9.5: We can cut a surface provided the orientation of the cut is given as above.

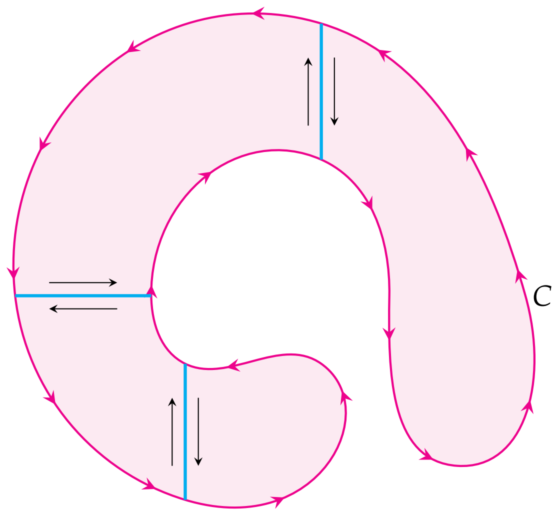


Figure 9.6: Multiple cuts can be made of a surface with orientation as above.

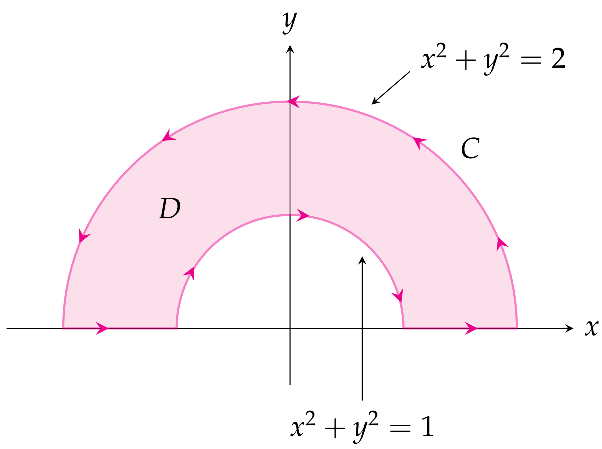


Figure 9.7: For Question 9.8.

9.3 Green's Theorem for regions with holes

Green's Theorem can be extended to apply to regions with holes. Suppose $\partial D_1 = C_1$ and $\partial D_2 = C_2$ then $D = D_1 - D_2$. Further assume C_1 and C_2 have opposite orientations with the "outer" boundary having positive orientation as in Figure 9.8. Divide D into D' and D'' as in Figure 9.9 and apply Green's Theorem:

$$\begin{aligned}
\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A &= \iint_{D'} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A + \iint_{D''} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A \\
&= \int_{\partial D'} P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{\partial D''} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{C_2} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y.
\end{aligned}$$

Note that the line integrals along the shared boundary in opposite directions cancel (Figure 9.10).

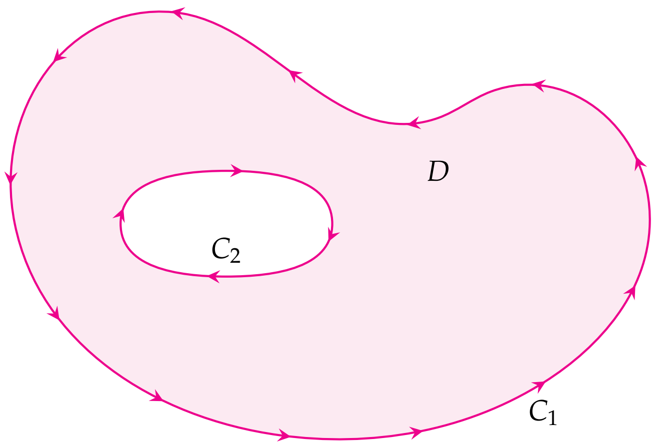


Figure 9.8: Setup for Green's Theorem proof.

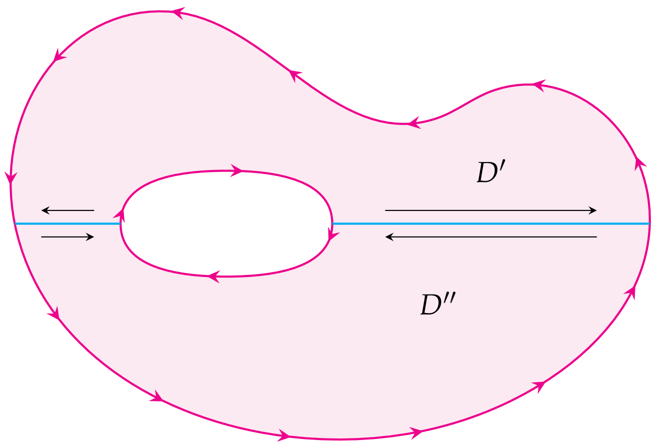


Figure 9.9: Setup for Green's Theorem proof.

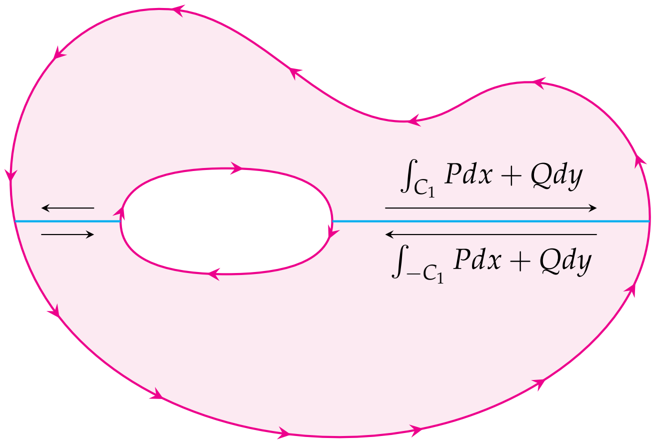


Figure 9.10: The reverse directions ensure the integrals cancel.

$$\int_{C_1} P\,dx + Q\,dy = -\int_{-C_1} P\,dx + Q\,dy$$
$$\implies \int_{C_1} P\,dx + Q\,dy + \int_{-C_1} P\,dx + Q\,dy = 0$$

9.4 End of lecture exercises

Question 9.9. Evaluate $\oint_C y^2 dx + x^2 y\, dy$ where C is the rectangle $(0,0)$, $(5,0)$, $(5,4)$, and $(0,4)$ directly and with Green's Theorem.

GREEN'S. We know ∂D is the rectangle so $D = [0,5] \times [0,4]$ and thus

$$\oint_C y^2 dx + x^2 y\, dy = \int_0^5 \int_0^4 \frac{\partial x^2 y}{\partial x} - \frac{\partial y^2}{\partial y}\, dy\, dx = \int_0^5 \int_0^4 2xy - 2y\, dy dx$$

$$= \int_0^5 \left[xy^2 - y^2\right]_0^4 dx = 16 \int_0^5 x - 1\, dx = 16 \left[\frac{x^2}{2} - x\right]_0^5 = 120.$$

◆

DIRECTLY. See Figure 9.11.

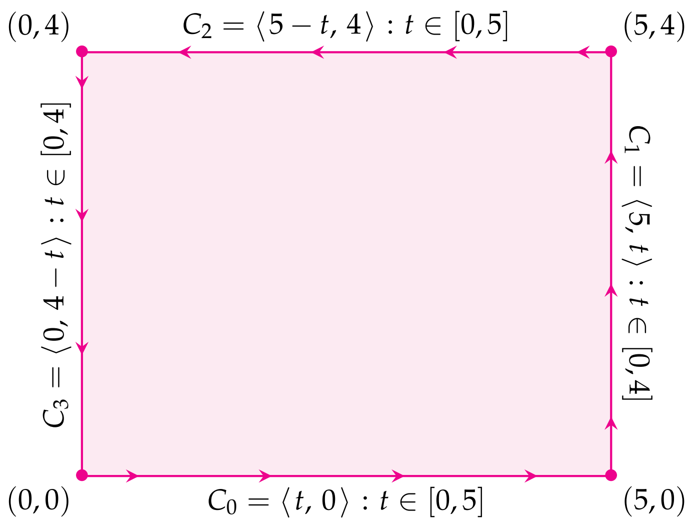


Figure 9.11: For Question 9.9.

C_0 $(x, y) = (t, 0) \implies (dx, dy) = (dt, 0)$ and so

$$\int_{C_0} y^2\, dx + x^2 y\, dy = \int_0^5 (0^2)\, dt + (t^2 \cdot 0y)\, d0 = 0.$$

C_1 $(x, y) = (5, t) \implies (dx, dy) = (0, dt)$ and so

$$\int_{C_1} y^2\, dx + x^2 y\, dy = \int_0^4 (t^2)\, d0 + (5^2 t)\, dt = \frac{25}{2}\left[t^2\right]_0^4 = 200.$$

C_2 $(x, y) = (5-t, 4) \implies (dx, dy) = (-dt, 0)$ and so

$$\int_{C_2} y^2\, dx + x^2 y\, dy = -\int_0^5 16 dt = -80.$$

C_3 $(x, y) = (0, 4-t) \implies (\mathrm{d}0, \mathrm{d}y) = (0, -\mathrm{d}t)$ and so

$$\int_{C_3} y^2\,\mathrm{d}x + x^2 y\,\mathrm{d}y = \int_0^4 (4-t)^2\,\mathrm{d}0 + (0^2 \cdot (4-t))\,\mathrm{d}0 = 0$$

Finally, combining answers we have

$$\int_C y^2\,\mathrm{d}x + x^2 y\,\mathrm{d}y = 0 + 200 - 80 + 0 = 120.$$

◆

Question 9.10. Evaluate $\oint_C xy\,\mathrm{d} + x^2y^3\,\mathrm{d}y$ where C is the triangle given by $(0,0)$, $(1,0)$, and $(1,2)$ directly and with Green's Theorem. See Figure 9.12.

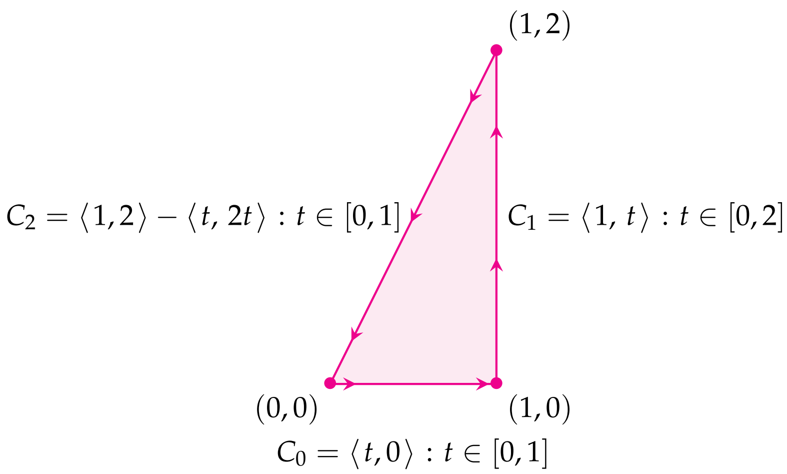


Figure 9.12: For Question 9.10.

GREEN'S. When $\partial D = C$ we have $D_{xy} = [0,1] \times [0,2x]$ and thus

$$\int_0^1 \int_0^{2x} \frac{\partial x^2y^3}{\partial x} - \frac{\partial xy}{\partial y}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_0^1 \int_0^{2x} 2xy^3 - x\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left[\frac{1}{2}xy^4 - xy\right]_0^{2x}\,\mathrm{d}x$$

$$= \int_0^1 \frac{1}{2}x(2x)^4 - 2x^2\,\mathrm{d}x = \int_0^1 8x^5 - 2x^2\,\mathrm{d}x$$

$$= 2\int_0^1 4x^5 - x^2\,\mathrm{d}x = 2\left[\frac{4}{6}x^6 - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3}.$$

C_0 $(x, y) = (t, 0) \implies (\mathrm{d}0, \mathrm{d}y) = (\mathrm{d}t, 0)$ and so

$$\int_{C_0} xy\,\mathrm{d}x + x^2y^3\,\mathrm{d}y = \int_0^1 (x \cdot 0)\,\mathrm{d}t + (t^2 \cdot 0)\,\mathrm{d}0 = 0.$$

C_1 $(x, y) = (1, t) \implies (\mathrm{d}x, \mathrm{d}y) = (0, \mathrm{d}t)$ and so

$$\int_{C_1} xy\,\mathrm{d}x + x^2y^3\,\mathrm{d}y = \int_0^2 (1\cdot t)\,\mathrm{d}0 + (1^2\cdot t^3)\,\mathrm{d}t = \int_0^2 t^3\,\mathrm{d}t = \left[\frac{t^4}{4}\right]_0^2 = 4.$$

C_2 $(x, y) = (1-t, 2(1-t)) \implies (\mathrm{d}x, \mathrm{d}y) = (-\mathrm{d}t, -2\mathrm{d}t)$ and so

$$\begin{aligned}
&\int_{C_2} xy\,\mathrm{d}x + x^2y^3\,\mathrm{d}y \\
&= -\int_0^1 2(1-t)(1-t)\,\mathrm{d}t + 16(1-t)^2(1-t)^3\,\mathrm{d}t \\
&= -2\int_0^1 (1-t)^2 + 8(1-t)^5\,\mathrm{d}t \\
&\text{Let } u = 1-t \implies \mathrm{d}u = -\mathrm{d}t \\
&= 2\int_1^0 u^2 + 8u^5\mathrm{d}u = -2\int_0^1 u^2 + 8u^5\mathrm{d}u \\
&= -2\left[\frac{u^3}{3} + 8\frac{u^6}{6}\right]_0^1 = -2\left(\frac{1}{3} + \frac{4}{3}\right) = -\frac{10}{3}.
\end{aligned}$$

Finally, combining answers gives $\int_C xy\,\mathrm{d}x + x^2y^3\,\mathrm{d}y = 0 + 4 - \frac{10}{3} = \frac{2}{3}$. ◆

Question 9.11. Evaluate $\oint_C y\,\mathrm{d}x - x\,\mathrm{d}y$ where C is the radius 2 circle centered at the origin directly and with Green's Theorem.

GREEN'S. $\partial D = C \implies D_{\theta r} = [0, 2\pi] \times [0, 2]$ and so

$$\oint_C y\,\mathrm{d}x - x\,\mathrm{d}y = \iint_D \frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y}\,\mathrm{d}A = \int_0^{2\pi}\int_0^2 -2\,r\,\mathrm{d}r\mathrm{d}\theta$$

$$= -2\int_0^{2\pi}\mathrm{d}\theta\int_0^2 r\mathrm{d}r = -2(2\pi)\left[\frac{r^2}{2}\right]_0^2 = -8\pi.$$

◆

DIRECT. Here we have

$$(x, y) = (2\cos\theta, 2\sin\theta) \implies (\mathrm{d}x, \mathrm{d}y) = (-2\sin\theta\,\mathrm{d}\theta, 2\cos\theta\,\mathrm{d}\theta)$$

and so

$$\oint_C y\,\mathrm{d}x + (-x)\mathrm{d}y = -4\int_0^{2\pi}\sin^2\theta + 4\cos^2\theta = -4\int_0^{2\pi}\mathrm{d}\theta = -8\pi.$$

◆

Question 9.12. Evaluate $\int_C y^3\,\mathrm{d}x - x^3\,\mathrm{d}y$ where C is the circle $x^2 + y^2 = 4$.

ANSWER. $\partial D = C \implies D_{\theta r} = [0, 2\pi] \times [0, 2]$ and so

$$\begin{aligned}
&\int_C y^3\,\mathrm{d}x - x^3\,\mathrm{d}y \\
&= \iint_D \frac{\partial - x^3}{\partial x} - \frac{\partial y^3}{\partial y}\,\mathrm{d}A = \iint_D -3x^2 - 3y^2\,\mathrm{d}A
\end{aligned}$$

$$= -3 \iint_D x^2 + y^2 \,\mathrm{d}A = -3 \int_0^{2\pi} \int_0^2 2\, r \,\mathrm{d}r \mathrm{d}\theta$$

$$= -3 \int_0^{2\pi} \mathrm{d}\theta \int_0^2 2\, r \,\mathrm{d}r = -3(2\pi - 0)(2^2 - 0^2) = -24\pi.$$

◆

Question 9.13. Evaluate $\oint_C (y + \mathrm{e}^{\sqrt{x}})\,\mathrm{d}x + (2x + \cos y^2)\,\mathrm{d}y$, where C is boundary of of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

ANSWER. $\partial D = C \implies D_{xy} = [0, 1] \times [x^2, \sqrt{x}]$.

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \,\mathrm{d}A = \iint_D 2 - 1 \,\mathrm{d}A = \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \,\mathrm{d}y\mathrm{d}x = \int_0^1 [y]_{x^2}^{\sqrt{x}}$$

$$= \int_0^1 \sqrt{x} - x^2 \,\mathrm{d}x = \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \right]_0^1 = \frac{1}{3}(2 - 1) = \frac{1}{3}.$$

◆

10 Curl and Divergence

We define two operations on *vector fields* that are fundamental to the study of fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a *vector field* whereas the other produces a *scalar field*.

Curl. Let $\boldsymbol{F}$ be a vector field. The *curl of* $\boldsymbol{F}$ is the *vector field* on $\mathbb{R}^3$ given by

$$\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F}.$$

Proposition 10.1. Let $\boldsymbol{F} = \langle P, Q, R \rangle$ then

$$\operatorname{curl} \boldsymbol{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

PROOF.

$$\begin{aligned} \operatorname{curl} \boldsymbol{F} &= \nabla \times \boldsymbol{F} \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \boldsymbol{k}. \end{aligned}$$

■

Question 10.2. Let $\boldsymbol{F}(x, y, z) = \langle xz, xyz, -y^2 \rangle$ and find $\operatorname{curl} \boldsymbol{F}$.

ANSWER.

$$\begin{aligned} \operatorname{curl} \boldsymbol{F} &= \left\langle \frac{\partial -y^2}{\partial y} - \frac{\partial xyz}{\partial z}, \frac{\partial xz}{\partial z} - \frac{\partial -y^2}{\partial x}, \frac{\partial xyz}{\partial x} - \frac{\partial xz}{\partial y} \right\rangle \\ &= \langle -2y - xy, x - 0, yz - 0 \rangle = \langle -y(2 + x), x, yz \rangle. \end{aligned}$$

◆

Recall $\mathbf{0} = \langle 0, \ldots, 0 \rangle$ and that $\boldsymbol{a} \times \boldsymbol{a} = \mathbf{0}$ for any vector $\boldsymbol{a}$. The following theorem is a type of analogue of this.

Theorem 10.3. If f is a function of three variables with continuous second-order partial derivatives then $\operatorname{curl}(\nabla f) = \mathbf{0}$.

PROOF.

$$\begin{aligned}\operatorname{curl}(\nabla f) &= \nabla \times (\nabla f) \\ &= \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle = \langle 0, 0, 0 \rangle .\end{aligned}$$

■

Conservative. A *conservative vector field* is one for which $\boldsymbol{F} = \nabla f$.

Corollary 10.4. $\boldsymbol{F}$ is conservative $\implies$ $\operatorname{curl} \boldsymbol{F} = \mathbf{0}$.

Corollary 10.5. $\operatorname{curl} \boldsymbol{F} \neq \mathbf{0} \implies \boldsymbol{F}$ is *not* conservative.

Question 10.6. Prove $\boldsymbol{F}(x, y, z) = \left\langle xz, xyz, -y^2 \right\rangle$ is *not* conservative.

ANSWER. We already calculated $\operatorname{curl} \boldsymbol{F} = \langle -y(2 + x), x, yz \rangle \neq \mathbf{0}$. Thus by the Theorem $\boldsymbol{F}$ is not conservative. ◆

Question 10.7. Let $\boldsymbol{F} = \langle P, Q, 0 \rangle$ where the x-y component is illustrated in Figure 10.1. Is the vector field conservative? If not, which direction does $\operatorname{curl} \boldsymbol{F}$ point?

ANSWER. Notice the vector field has the form $\boldsymbol{F} = \langle 0, Q(y), 0 \rangle$ where Q is an *decreasing* function. Thereby

$$\operatorname{curl} \boldsymbol{F} = \left\langle \frac{\partial 0}{\partial y} - \frac{\partial Q(y)}{\partial z}, \frac{\partial 0}{\partial z} - \frac{\partial 0}{\partial x}, \frac{\partial Q(y)}{\partial x} - \frac{\partial 0}{\partial y} \right\rangle = \langle 0, 0, 0 \rangle .$$

◆

Question 10.8. Let $\boldsymbol{F} = \langle P, Q, 0 \rangle$ where the x-y component is illustrated in Figure 10.2. Is the vector field conservative? If not, which direction does $\operatorname{curl} \boldsymbol{F}$ point?

ANSWER. Notice the vector field has form $\boldsymbol{F} = \langle P(y), 0, 0 \rangle$ where $P(y)$ is a *increasing* function. Thereby $\operatorname{curl} \boldsymbol{F} = \left\langle \frac{\partial 0}{\partial y} - \frac{\partial 0}{\partial z}, \frac{\partial P(y)}{\partial z} - \frac{\partial 0}{\partial x}, \frac{\partial 0}{\partial x} - \frac{\partial P(y)}{\partial y} \right\rangle = \left\langle 0, 0, -\frac{\partial P(y)}{\partial y} \right\rangle = \langle 0, 0, k \rangle$ for $k \in \mathbb{R}^{>0}$. ◆

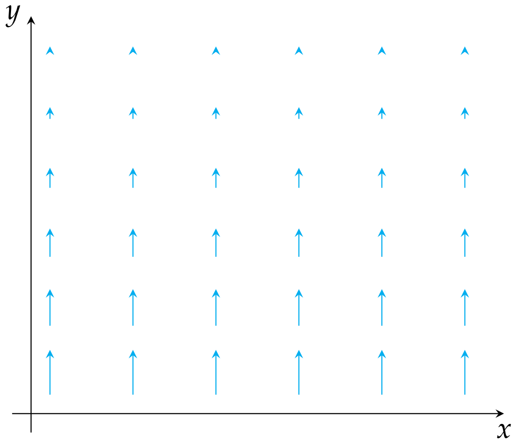


Figure 10.1: For Question 10.7 and 10.13.

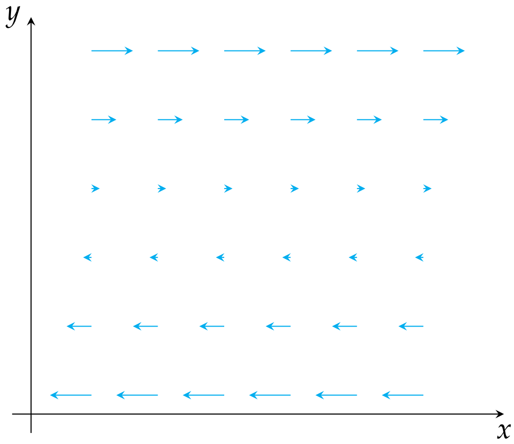


Figure 10.2: For Question 10.8 and 10.14.

Question 10.9. Prove $\boldsymbol{F} = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle$ is conservative and then find f such that $\boldsymbol{F} = \nabla f$.

ANSWER.

$$\operatorname{curl} F = \left\langle \frac{\partial\, 3xy^2z^2}{\partial y} - \frac{\partial\, 2xyz^3}{\partial z}, \frac{\partial\, y^2z^3}{\partial z} - \frac{\partial\, 3xy^2z^2}{\partial x}, \frac{\partial\, 2xyz^3}{\partial x} - \frac{\partial\, y^2z^3}{\partial y} \right\rangle$$
$$= \left\langle 6xyz^2 - 6xyz^2, 3y^2z^2 - 3y^2z^2, 2yz^3 - 2yz^3 \right\rangle = \langle 0, 0, 0 \rangle = \boldsymbol{0}$$

and therefore the $\boldsymbol{F}$ is conservative.

To find f consider that we want

$$\frac{\partial f}{\partial x} = y^2z^3, \qquad \frac{\partial f}{\partial y} = 2xyz^3, \qquad \frac{\partial f}{\partial z} = 3xy^2z^2$$

and so

$$f = \int \frac{\partial f}{\partial x}\, \mathrm{d}x = xy^2z^3 + g(y,z)$$
$$\implies \frac{\partial f}{\partial y} = 2xyz^3 + \frac{\partial g(y,z)}{\partial y} \implies \frac{\partial g(y,z)}{\partial y} = 0$$
$$\implies f = xy^2z^3 + h(z) \implies \frac{\partial f}{\partial z} = 3xy^2z^2 + \frac{\mathrm{d}h(z)}{\mathrm{d}z}$$
$$\implies h(z) = K \text{ for some } K \in \mathbb{R}.$$

Thus $f = 3xy^2 + z^2 + K \implies \nabla f = \boldsymbol{F}$. ◆

This operation is called the *curl* because it is associated with rotations. For instance, suppose $\boldsymbol{F}$ represents velocity in a fluid flow, then particles near (x,y,z) tend to *rotate* or *curl around* vector $\operatorname{curl} \boldsymbol{F}(x,y,z)$. The magnitude of the vector indicates how quickly the particles do so (Figure 10.3). If $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$ at a point P, then P is free from rotation at P and $\boldsymbol{F}$ is called *irrotatoinal* at p. In other words, there is no "whirlpool" or "eddy" at P.

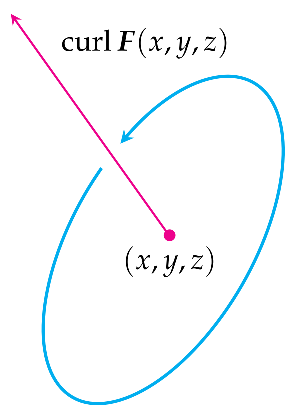


Figure 10.3: The curl.

10.1 Divergence

Divergence. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, then the *divergence of* $\boldsymbol{F}$ is

$$\operatorname{div} \boldsymbol{F} := \nabla \cdot \boldsymbol{F}$$

provided the required partial derivatives exist.

Notice $\operatorname{curl} \boldsymbol{F}$ is a vector field whereas $\operatorname{div} \boldsymbol{F}$ is a scalar field.

Question 10.10. Let $\boldsymbol{F} = \langle xz, xyz, -y^2 \rangle$ and find $\operatorname{div} \boldsymbol{F}$.

ANSWER. $\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \dfrac{\partial xz}{\partial x} + \dfrac{\partial xyz}{\partial y} + \dfrac{\partial -y^2}{\partial z} = z + xz + 0.$ ◆

Theorem 10.11. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field whose components have second-order partial derivatives:

$$\operatorname{div} \operatorname{curl} \boldsymbol{F} = 0. \tag{10.1}$$

PROOF.

$$\begin{aligned}
\operatorname{div} \operatorname{curl} \boldsymbol{F} &= \nabla \cdot (\nabla \times \boldsymbol{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial y} + \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 R}{\partial x \partial y} + \frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial^2 P}{\partial x \partial y} \\
&= 0.
\end{aligned}$$

■

Question 10.12. Show there is no vector field $\boldsymbol{G}$ such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$ when $\boldsymbol{F} = \langle xz, xyz, -y^2 \rangle$.

ANSWER. We have $\operatorname{div} \boldsymbol{F} = z + xz + 0 \implies \operatorname{div} \boldsymbol{F} \neq 0$. *Towards a contradiction* suppose $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$.

$$\begin{aligned}
&\boldsymbol{F} = \operatorname{curl} \boldsymbol{G} \\
&\qquad \implies \operatorname{div} \boldsymbol{F} = \operatorname{div} \operatorname{curl} \boldsymbol{G} = 0 && \text{By Equation (10.1).} \\
&\qquad \implies \operatorname{div} \boldsymbol{F} \neq 0 \wedge \operatorname{div} \boldsymbol{F} \neq 0 \ \lightning
\end{aligned}$$

(↯ is the symbol for "contradiction"). Thereby it must be the case that there is no $\boldsymbol{G}$ such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$. ◆

Divergence can be understood in the context of fluid flow as well. If $\boldsymbol{F}(x, y, z)$ is the velocity of a fluid then $\operatorname{div} \boldsymbol{F}(x, y, z)$ is the *net rate of change of the mass of fluid flowing from* (x, y, z) *per unit volume*. That is, $\operatorname{div} \boldsymbol{F}(x, y, z)$ is a measure of the tendency of the fluid to *diverge* from (x, y, z).

Incomprehensible. When $\operatorname{div} \boldsymbol{F} = 0$ then $\boldsymbol{F}$ is said to be *incompressible*.

Question 10.13. Let $\boldsymbol{F} = \langle P, Q, 0 \rangle$ where the x-y component is illustrated in Figure 10.7. Is $\operatorname{div} \boldsymbol{F}$ positive, negative, or zero?

ANSWER. Notice the vector field has the form $\boldsymbol{F} = \langle 0, Q(y), 0 \rangle$ where Q is an *decreasing* function. Thereby

$$\frac{\partial 0}{\partial x} + \frac{\partial Q(y)}{\partial y} + \frac{\partial 0}{\partial z} = \frac{\partial Q}{\partial y} < 0.$$

◆

Question 10.14. Let $\boldsymbol{F} = \langle P, Q, 0 \rangle$ where the x-y component is illustrated in Figure 10.8. Is $\operatorname{div} \boldsymbol{F}$ positive, negative, or zero?

ANSWER. Notice the vector field has form $\boldsymbol{F} = \langle P(y), 0, 0 \rangle$ where $P(y)$ is a *increasing* function. Thereby

$$\frac{\partial P(y)}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 0}{\partial z} = 0.$$

◆

10.2 Laplace Operator

Recall Laplace's equation $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ and note that we can rewrite this equation as $\nabla = \nabla \cdot (\nabla f) = \operatorname{div}(\nabla f)$.

Laplace operator. $\nabla^2 = \nabla \cdot \nabla$ is called the *Laplace operator*.

We can also apply the Laplace operator ∇^2 to a vector field:

$$\nabla^2 \boldsymbol{F} = \left\langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \right\rangle.$$

10.3 Vector Forms of Green's Theorem

The curl and divergence operators allow for a rewriting of Green's Theorem that is useful later.

Green's Theorem (Curl Form). Let D be a planar region such that $C = \partial D$ and suppose $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}$ satisfies the hypothesis of Green's Theorem. Then

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_D (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{k} \, \mathrm{d}A$$

where $\boldsymbol{k} = \langle 0, 0, 1 \rangle$.

PROOF. Let $\boldsymbol{F} = \langle P, Q, 0 \rangle$ so that

$$\operatorname{curl} \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \boldsymbol{k}.$$

Therefore $(\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \boldsymbol{k} \cdot \boldsymbol{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ and so

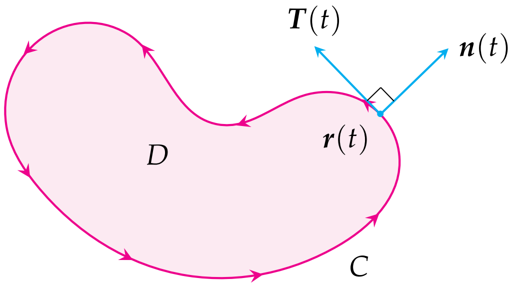
$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, \mathrm{d}A = \iint_D (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{k} \, \mathrm{d}A$$

and the result follows. ■

Green's Theorem (Div form). Assume the hypothesis of Green's Theorem. Then

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}s = \iint_D \operatorname{div} \boldsymbol{F}(x,y) \, \mathrm{d}A.$$

where $\boldsymbol{T}(t)$ is the *tangent* of $\boldsymbol{r}(t)$ and $\boldsymbol{n}(t)$ the normal.



10.4 End of lecture exercises

Question 10.15. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field. Are the following meaningful? If so, state whether the result is a *scalar field* or *vector field*.

1. ∇f,
2. $\operatorname{curl} \nabla f$,
3. $\nabla \operatorname{div} \boldsymbol{F}$,
4. $\nabla \operatorname{div} f$,
5. $\operatorname{div} \operatorname{div} \boldsymbol{F}$,
6. $\operatorname{div} \operatorname{curl} \nabla f$.

ANSWER. Vector field. Vector field. Vector field. Meaningless. Meaningless. Scalar field. ◆

Question 10.16. Find the divergence and curl of

$$\boldsymbol{F} = \langle P, Q, R \rangle = \left\langle xy^2z^2, x^2yz^2, x^2y^2z \right\rangle .$$

ANSWER.

$$\begin{aligned}\operatorname{curl} \boldsymbol{F} &= \left\langle \frac{\partial\, x^2y^2z}{\partial y} - \frac{\partial\, x^2yz^2}{\partial z}, \frac{\partial\, xy^2z^2}{\partial z} - \frac{\partial\, x^2y^2z}{\partial x}, \frac{\partial\, x^2yz^2}{\partial x} - \frac{\partial\, xy^2z^2}{\partial y} \right\rangle \\ &= \left\langle 2x^2yz - 2x^2yz, 2xy^2z - 2xy^2z, 2xyz^2 - 2xyz^2 \right\rangle = \langle 0, 0, 0 \rangle .\end{aligned}$$

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial\, xy^2z^2}{\partial x} + \frac{\partial\, x^2yz^2}{\partial y} + \frac{\partial x^2y^2z}{\partial z} = y^2z^2 + x^2z^2 + x^2y^2.$$

◆

Question 10.17. Find the divergence and curl of

$$\boldsymbol{F} = \langle \ln(2y + 3z), \ln(x + 3z), \ln(x + 2y) \rangle .$$

ANSWER.

$$\begin{aligned}\operatorname{curl} \boldsymbol{F} = \Bigg\langle &\frac{\partial\, \ln(x + 2y)}{\partial y} - \frac{\partial\, \ln(x + 3z)}{\partial z}, \\ &\frac{\partial\, \ln(2y + 3z)}{\partial z} - \frac{\partial \ln(x + 2y)}{\partial x}, \frac{\partial \ln(x + 3z)}{\partial x} - \frac{\partial \ln(2y + 3z)}{\partial y} \Bigg\rangle.\end{aligned}$$

$$\operatorname{div} \boldsymbol{F} = \frac{\partial \ln(2y + 3z)}{\partial x} + \frac{\partial \ln(x + 3z)}{\partial y} + \frac{\partial \ln(x + 2y)}{\partial z}.$$

◆

Question 10.18. If $\boldsymbol{F} = \langle 1, \sin z, y\cos z\rangle$ is a conservative find $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\boldsymbol{F} = \nabla f$.

ANSWER. $\boldsymbol{F}$ is conservative because

$$\operatorname{curl} \boldsymbol{F} = \left\langle \frac{\partial\, y\cos z}{\partial y} - \frac{\partial\, \sin z}{\partial z}, \frac{\partial\, 1}{\partial z} - \frac{\partial\, y\cos z}{\partial x}, \frac{\partial\, \sin z}{\partial x} - \frac{\partial\, 1}{\partial y} \right\rangle = 0.$$

To find f so that $\nabla f = \boldsymbol{F}$ consider

$$(1)\ \frac{\partial f}{\partial x} = 1, \qquad (2)\ \frac{\partial f}{\partial y} = \sin z, \qquad (3)\ \frac{\partial f}{\partial z} = y\cos z.$$

$$\begin{aligned}
(1) &\implies \int \frac{\partial f}{\partial x}\,\mathrm{d}x = \int 1\,\mathrm{d}x \implies f = x + g(y,z) \\
&\implies \frac{\partial f}{\partial y} = 0 + \frac{\partial g(y,z)}{\partial y} \overset{(2)}{=} \sin z \implies \int \frac{\partial\, g(y,z)}{\partial y}\,\mathrm{d}y = \int \sin z\,\mathrm{d}y \\
&\implies g(y,z) = y\sin z + h(z) \qquad (*)
\end{aligned}$$

$$\begin{aligned}
(*) &\implies f = x + y\sin z + h(z) \implies \frac{\partial f}{\partial z} = 0 + y\cos z + \frac{\partial h}{\partial z} \overset{(3)}{=} y\cos z \\
&\implies \frac{\partial h}{\partial z} = 0 \implies \int \frac{\partial h}{\partial z}\,\mathrm{d}z = \int 0\,\mathrm{d}z \implies h = K \text{ for } K \in \mathbb{R}.
\end{aligned}$$

Therefore $f = x + y\sin z + K$ for $K \in \mathbb{R}$ and $\nabla f = \langle 1, \sin z, y\cos z\rangle$. ◆

11 Surface Integrals

The relationship between *surface integrals* and *surface area* is analogous to the relationship between *line integrals* and *arc length*. We start with parametric surface's S and then extend to the special case where S is the graph of a function of two variables. Recall a parametric surface S has *vector equation* $\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\boldsymbol{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle .$$

Lastly, we use the following notation for this lecture

$$\boldsymbol{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = \left\langle \frac{\partial P}{\partial u}, \frac{\partial Q}{\partial u}, \frac{\partial R}{\partial u} \right\rangle .$$

11.1 Parametric Surfaces

Surface Integral. Let S be given parametrically by $\boldsymbol{r}(u,v) : \mathbb{R}^2 \to \mathbb{R}^3$. The *surface integral of f over the parametric surface S* is

$$\iint_S f(x,y,z)\,\mathrm{d}S = \lim_{m,n\to\infty} \sum_{i=0}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}.$$

See Figure 11.1 and 11.2.

The following allows for the computation of surface integrals via conversion to a double integral over (not necessarily rectangular) D.

Parametric Surface Integral. Let $S = \{\boldsymbol{r}(u,v) : (u,v) \in D\}$ then

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\boldsymbol{r}(u,v))\,||\boldsymbol{r}_u \times \boldsymbol{r}_v||\,\mathrm{d}A. \tag{11.1}$$

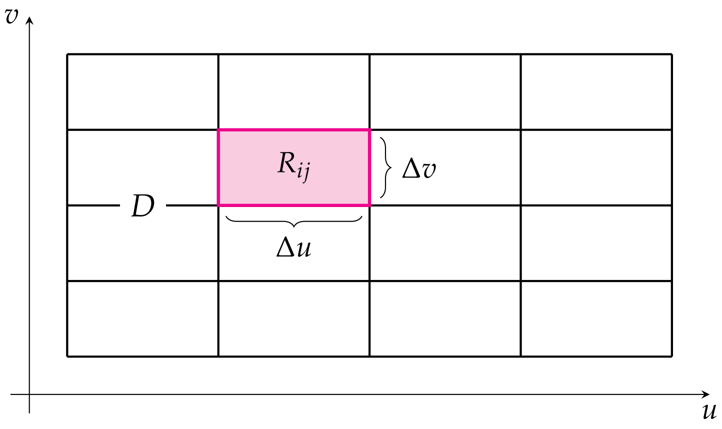


Figure 11.1: The so-called "domain space".

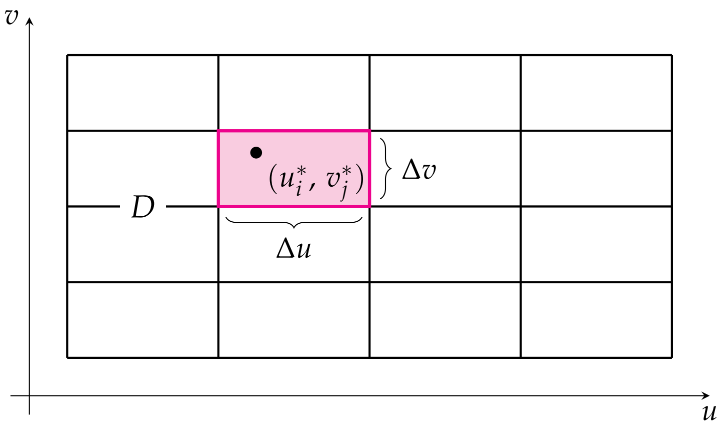


Figure 11.2: Choosing a s ample point from the domain space.

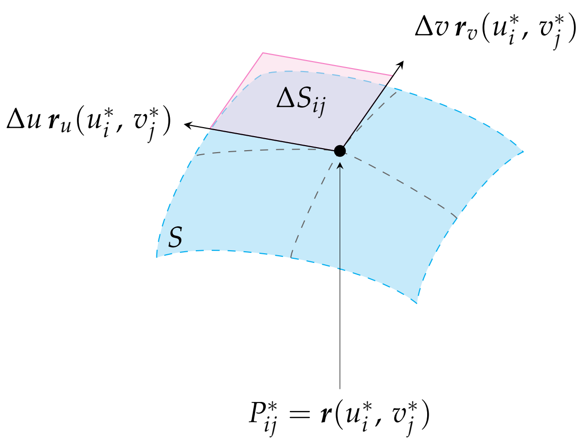


Figure 11.3: If $\boldsymbol{r}_u^* := \boldsymbol{r}_u(u_i^*, v_j^*)$ and $\boldsymbol{r}_v^* := \boldsymbol{r}_v(u_i^*, v_j^*)$ then

$$\begin{aligned} \Delta S_{ij} &\approx |\Delta u\, \boldsymbol{r}_u^* \times \Delta v\, \boldsymbol{r}_v^*| = |\boldsymbol{r}_u^* \times \boldsymbol{r}_v^*|\, \Delta u\, \Delta v \\ &\implies f(P_{ij}^*)\Delta S_{ij} \approx f(\boldsymbol{r}(u_i^*, v_j^*))\, |\boldsymbol{r}_u^* \times \boldsymbol{r}_v^*|\, \Delta u\, \Delta v. \end{aligned}$$

Notice this proposition is similar to the formula for line integrals:

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(\boldsymbol{r}(t))\,|\boldsymbol{r}'(t)|\,\mathrm{d}t.$$

Proposition 11.1.

$$\iint_S 1\,\mathrm{d}S = \iint_D |\boldsymbol{r}_u \times \boldsymbol{r}_v|\,\mathrm{d}A = \text{SurfaceArea}(S).$$

PROOF. Substitute $f(x,y,z) = 1$ in the previous proposition. ■

Question 11.2. Compute the surface integral $\iint_S x^2\,\mathrm{d}S$ where S is the *unit sphere*.

ANSWER. In spherical the unit sphere is given by

$$\langle x,\, y,\, z\rangle \leftarrow \langle \sin\phi\cos\theta,\, \sin\phi\sin\theta,\, \cos\phi\rangle : \phi \in [0,\, \pi] \wedge \theta \in [0,\, 2\pi]$$

which means $\boldsymbol{r}(u,v) = \boldsymbol{r}(\phi,\, \theta) = \langle \sin\phi\cos\theta,\, \sin\phi\sin\theta,\, \cos\phi\rangle$ is the parameterization of S.

Recall $\boldsymbol{r} = \langle P,\, Q,\, R\rangle = \langle \sin\phi\cos\theta,\, \sin\phi\sin\theta,\, \cos\phi\rangle$ and so

$$\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial P}{\partial \phi} & \frac{\partial Q}{\partial \phi} & \frac{\partial R}{\partial \phi} \\ \frac{\partial P}{\partial \theta} & \frac{\partial Q}{\partial \theta} & \frac{\partial R}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= \left\langle 0 + \sin^2\phi\cos\theta,\, \sin^2\phi\sin\theta - 0,\, \cos\phi\sin\phi\cos^2\theta + \cos\phi\sin\phi\sin^2\theta \right\rangle$$
$$= \left\langle \sin^2\phi\cos\theta,\, \sin^2\phi\sin\theta,\, \cos\phi\sin\phi \right\rangle$$

$$||\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta|| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \cos^2\phi\sin^2\phi}$$
$$= \sqrt{\sin^4\phi + \cos^2\phi\sin^2\phi} = \sqrt{\sin^2\phi\,(\sin^2\phi + \cos^2\phi)} = \sqrt{\sin^2\phi} = \sin\phi.$$

By Proposition 11.1 we have

$$\iint_S x^2\,\mathrm{d}S = \iint_D (\sin\phi\cos\theta)^2\,|\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta|\,\mathrm{d}A = \iint_D \sin^3\phi\cos^2\theta\,\sin\phi\,\mathrm{d}A$$
$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta\,\mathrm{d}\phi\,\mathrm{d}\theta = \int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta \int_0^{\pi}\sin^3\phi\,\mathrm{d}\phi$$
$$= \int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,\mathrm{d}\theta \int_0^{\pi}(\sin\phi - \sin\phi\cos^2\phi)\,\mathrm{d}\phi$$
$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi} = \frac{4\pi}{3}.$$

◆

11.2 Graphs

Any *surface* S given by $z = g(x,y)$ can be regarded as the parameterized surface

$$S = \{\boldsymbol{r}(u,v) = \langle u,\ v,\ g(u,v)\rangle : (u,v) \in D\}$$

with

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & \frac{\partial g}{\partial u} \\ 0 & 1 & \frac{\partial r}{\partial v} \end{vmatrix} = \left|\left\langle -\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1 \right\rangle\right| = \sqrt{\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + 1}\,.$$

Graph Surface Integral. When $S = \{(u,v,g(u,v)) : (u,v) \in D\}$ for $g : \mathbb{R}^2 \to \mathbb{R}$ then

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(u,v,g(u,v))\sqrt{\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + 1}\ \mathrm{d}A. \tag{11.2}$$

We have similar equations for when S is given by $y = h(x,z)$ or $x = h(y,z)$. For instance the former admits

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(u,h(u,v),v)\sqrt{\left(\frac{\partial h}{\partial u}\right)^2 + \left(\frac{\partial h}{\partial v}\right)^2 + 1}\ \mathrm{d}A.$$

Question 11.3. Evaluate $\iint_S y\,\mathrm{d}S$ where S is the surface given by

$$z = x + y^2 : (x,y) \in [0,1] \times [0,2].$$

ANSWER. Here our surface S is given by $g(u,v) = u + v^2$ and so by Equation 11.2 we have

$$\begin{aligned}
\iint_S y\ \mathrm{d}S &= \iint_D v\sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}\ \mathrm{d}A \\
&= \iint_D v\sqrt{1 + (1)^2 + (2v)^2}\ \mathrm{d}A = \int_0^1 \int_0^2 v\sqrt{1+1+4v^2}\ \mathrm{d}v\,\mathrm{d}u \\
&= \int_0^1\ \mathrm{d}u \int_0^2 v\sqrt{2(1+v^2)} = \int_0^1\ \mathrm{d}u\,\sqrt{2}\int_0^2 v\sqrt{1+2v^2}\ \mathrm{d}v \\
&= (1-0)\,\sqrt{2}\left[\frac{1}{4}\frac{2}{3}\left(1+2v^2\right)^{\frac{3}{2}}\right]_0^2 = \frac{13\sqrt{2}}{3}.
\end{aligned}$$

◆

Proposition 11.4. Is $S = S_1 \cup \cdots S_n$ is a piecewise-smooth surface that intersects only at boundaries then

$$\iint_{S_1\cup\cdots\cup S_n} f(x,y,z)\,\mathrm{d}S = \iint_{S_1} f(x,y,z)\,\mathrm{d}S + \cdots + \iint_{S_n} f(x,y,z)\,\mathrm{d}S.$$

Question 11.5. Evaluate $\iint_S z\,\mathrm{d}S$ where S is the surface given by

$S_1:$	$x^2+y^2=1$	Sides
$S_2:$	$x^2+y^2\leq 1$ at $z=0$	Base
$S_3:$	$z=1+x$	Lid

ANSWER. The answer is the sum of the following three integrals.

S_1 Since S_1 is a cylinder we use the cylindrical coordinate system to parameterize it by

$$\boldsymbol{r}(\theta,z) = \langle\cos\theta,\ \sin\theta,\ z\rangle : [\theta,\ z] \in [0,\ 2\pi]\times[0, 1+\cos\theta].$$

and so

$$\left|\frac{\partial \boldsymbol{r}}{\partial\theta}\times\frac{\partial \boldsymbol{r}}{\partial z}\right| = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = |\langle\cos\theta,\ \sin\theta,\ 0\rangle| = \sqrt{\cos^2\theta+\sin^2\theta} = 1.$$

Thereby

$$\begin{aligned}\iint_{S_1} z\,\mathrm{d}S &= \iint_D z\left|\boldsymbol{r}_\theta\times\frac{\partial \boldsymbol{r}}{\partial z}\right|\,\mathrm{d}A = \iint_D z\cdot 1\,\mathrm{d}z\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,\mathrm{d}z\,\mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} 1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\,\mathrm{d}\theta \\ &= \frac{1}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2}\end{aligned}$$

S_2 Here $z=0$ and so $\iint_{S_2} z\,\mathrm{d}S = \iint_{S_2} = 0$.

S_3 S_3 lies above the unit disk ($z\geq 0$ and $x^2+y^2\leq 1$) and on $z=1+x$. Thereby

$$\begin{aligned}\iint_{S_3} z\,\mathrm{d}S &= \iint_D (1+x)\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}A \\ &= \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{1+1+0}\;r\,\mathrm{d}r\mathrm{d}\theta \quad \text{(Convert to polar.)} \\ &= \sqrt{2}\int_0^{2\pi}\int_0^1 (r+r^2\cos\theta)\,\mathrm{d}r\mathrm{d}\theta = \sqrt{2}\int_0^{2\pi}\left[\frac{r^2}{2}+\frac{r^3}{3}\cos\theta\right]_0^1\mathrm{d}\theta\end{aligned}$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{2} + \frac{1}{3}\cos\theta \, d\theta = \sqrt{2}\left[\frac{\theta}{2} + \frac{1}{3}\sin\theta\right]_0^{2\pi} = \sqrt{2}\pi.$$

FINAL ANSWER $\iint_S z \, dS = \frac{3\pi}{2} + 0 + \sqrt{2}\pi = \frac{(3+2\sqrt{2})\pi}{2}.$ ◆

11.3 Oriented Surfaces

Suppose you are walking along the surface of the Möbius strip. Which way is up? Can you paint the Möbius strip two colours?

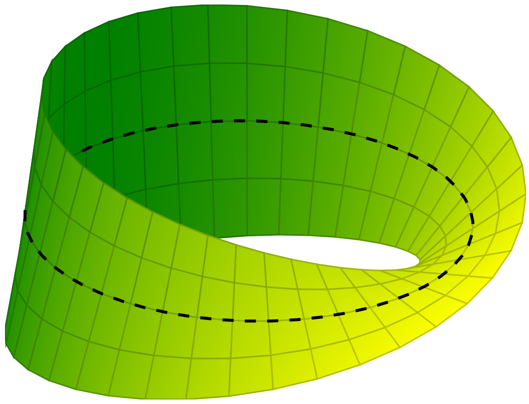


Figure 11.4: The Möbius strip.

The Möbius strip only has one side — it is *unorientable*. An *oriented surface* has two sides (as in two sides of a piece of paper or the outside and inside of a sphere).

Oriented. A surface S is *oriented* when there is a normal $\boldsymbol{n}$ at each $(x, y, z) \in S - \partial S$ and $\boldsymbol{n}$ varies continuously over S.

Because at each point there are *two* normals: $\boldsymbol{n}$ and $-\boldsymbol{n}$ the surface S is orientable in *two ways*. (That is, we must specify which normal is "up").

Proposition 11.6. Suppose the surface S is given by $z = g(x, y)$. The *unit normal* of g at (x, y) is given by

$$\hat{\boldsymbol{n}} = \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{1 + (g_x)^2 + (g_y)^2}}$$

(The z-component 1 indicates *positive orientation*).

PROOF. S is parameterized by $\boldsymbol{r} = \langle x, y, g(x, y)\rangle$ so

$$\boldsymbol{n} = \boldsymbol{r}_x \times \boldsymbol{r}_y = \langle 1, 0, g_x\rangle \times \langle 0, 1, g_y\rangle = \langle -g_x, -g_y, 1\rangle$$

and normalizing gives the result. ■

Positive Orientation. If S is a *smooth orientable surface* given parametrically by $\boldsymbol{r}(u, v)$ then the direction of *positive orientation* is given by

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}.$$

Question 11.7. What is the orientation induced by the sphere $x^2 + y^2 + z^2 = a^2$?

ANSWER. This sphere is parameterized by

$$\boldsymbol{r}(\phi, \theta) = \left\langle a^2 \sin^2 \phi \cos \theta,\ a^2 \sin^2 \phi \sin \theta,\ a^2 \sin \phi \cos \phi \right\rangle$$

where $|\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta| = a^2 \sin \phi$. Thus, the orientation induced by $\boldsymbol{r}(\phi, \theta)$ is given by

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta}{|\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta|} = \langle \sin \phi \cos \theta,\ \sin \phi \sin \theta,\ \cos \phi \rangle = \frac{1}{a} \boldsymbol{r}(\phi, \theta).$$

◆

Convention. For a *closed surface* (a surface that bounds a solid region) the *positive orientation* is the one for which the normals point *outward* from E.

11.4 Vector Fields

Flux. Let S be an *oriented surface* with unit normal $\boldsymbol{n}$ and $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector field such that $\operatorname{dom}(\boldsymbol{F}) \subseteq S$. The *surface integral of* $\boldsymbol{F}$ *over* S is

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S.$$

This integral is called the *flux of* $\boldsymbol{F}$ *across* S.

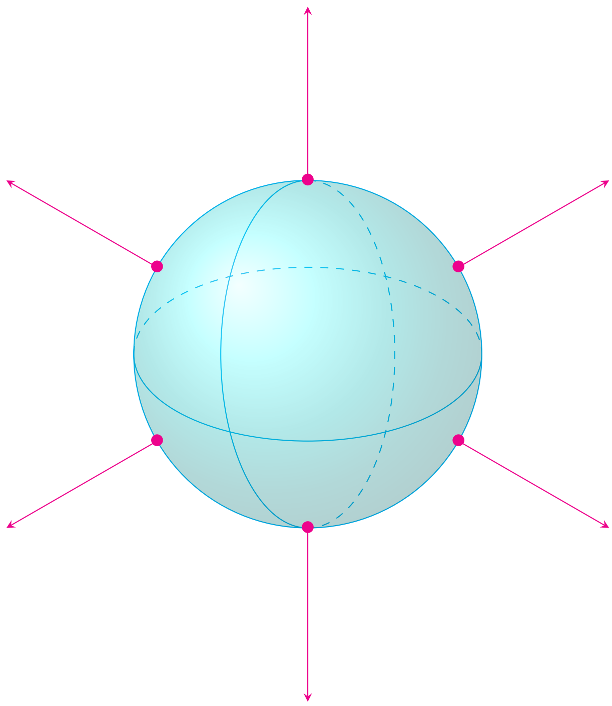


Figure 11.5: Positive orientation.

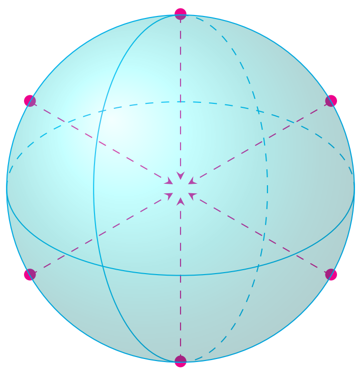


Figure 11.6: Negative orientation.

Parametric Flux. Let S be an *oriented surface* with *upward orientation* given by $S = \{\boldsymbol{r}(u,v) : (u,v) \in D\}$ then

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D \boldsymbol{F} \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v) \, \mathrm{d}A \qquad (11.3)$$

PROOF. Suppose $S = \{\boldsymbol{r}(u,v) : (u,v) \in D\}$ then

$$\begin{aligned}\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} &= \iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S \\ &= \iint_S \boldsymbol{F} \cdot \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} \, \mathrm{d}S \\ &= \iint_D \left(\boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} \right) |\boldsymbol{r}_u \times \boldsymbol{r}_v| \, \mathrm{d}A. \qquad \text{By definition.}\end{aligned}$$

Notice $\boldsymbol{F} \cdot \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}$ is a *scalar function* $f(x,y,z)$. ■

Question 11.8. Find the *flux* of $\boldsymbol{F}(x,y,z) = \langle z, \, y, \, x \rangle$ across the unit sphere.

ANSWER. The unit sphere is parameterized by

$$\boldsymbol{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \, \sin\phi\sin\theta, \, \cos\phi \rangle : (\phi,\theta) \in [0,\pi] \times [0,2\pi].$$

and thereby

$$\boldsymbol{F}(\boldsymbol{r}(\phi,\theta)) = \langle \cos\phi, \, \sin\phi\sin\theta, \, \sin\phi\cos\theta \rangle .$$

Recall $\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = \left\langle \sin^2\phi\cos\theta, \, \sin^2\phi\sin\theta, \, \sin\phi\cos\phi \right\rangle$ and thus

$$\begin{aligned}&\boldsymbol{F}(\boldsymbol{r}(\phi,\theta)) \cdot \boldsymbol{r}_\phi \times \boldsymbol{r}_\theta \\ &\quad = \langle \cos\phi, \, \sin\phi\sin\theta, \, \sin\phi\cos\theta \rangle \cdot \left\langle \sin^2\phi\cos\theta, \, \sin^2\phi\sin\theta, \, \sin\phi\cos\phi \right\rangle \\ &\quad = \cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta + \cos\phi\sin^2\phi\cos\theta \\ &\quad = 2\cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta.\end{aligned}$$

The flux (by definition) is given by

$$\begin{aligned}&\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} \\ &\quad = \iint_D \boldsymbol{F} \cdot (\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta) \, \mathrm{d}A \\ &\quad = \int_0^{2\pi} \int_0^{\pi} 2\cos\phi\sin^2\phi\cos\theta + \sin^3\phi\sin^2\theta \; \mathrm{d}\phi \, \mathrm{d}\theta \\ &\quad = \int_0^{2\pi} \int_0^{\pi} 2\cos\phi\sin^2\phi\cos\theta \; \mathrm{d}\phi \, \mathrm{d}\theta + \int_0^{2\pi} \int_0^{\pi} \sin^3\phi\sin^2\theta \; \mathrm{d}\phi \, \mathrm{d}\theta \\ &\quad = 2\int_0^{2\pi} \cos\theta \, \mathrm{d}\theta \int_0^{\pi} \cos\phi\sin^2\phi \, \mathrm{d}\phi + \int_0^{2\pi} \sin^2\theta \, \mathrm{d}\theta \int_0^{\pi} \sin^3\phi \, \mathrm{d}\phi\end{aligned}$$

$$= 0 + \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi} \left[-\frac{(2+\sin^2\theta)\cos\theta}{3}\right]_0^{\pi} = \frac{4\pi}{3}.$$

Note the zero arises because $\int_0^{2\pi} \cos\theta \, d\theta = 0$. ◆

Graph Flux. Let S be an *oriented surface* with *upward orientation* given by $S = \{(x, y, g(x,y)) : (x,y) \in D\}$ then

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D \boldsymbol{F} \cdot \langle -g_x, -g_y, 1 \rangle \, dA. \tag{11.4}$$

Negative orientation is obtained by negating the answer.

PROOF. S can be parameterized by $\boldsymbol{r}(x,y) = \langle x, y, g(x,y) \rangle$ and so

$$\boldsymbol{F} \cdot \langle \boldsymbol{r}_x \times \boldsymbol{r}_y \rangle = \boldsymbol{F} \cdot \langle -g_x, -g_y, 1 \rangle.$$

The result follows. ■

Question 11.9. What is $\iint_S \langle y, x, z \rangle \cdot d\boldsymbol{S}$ when S is the boundary of the *solid region* given by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$? See Figure 11.9.

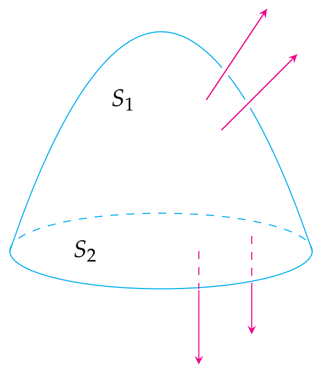


Figure 11.7: For Question 11.9

ANSWER. We perform the integral by breaking S into its top S_1 and bottom S_2. Since E is a closed surface we use the convention of positive (outward) orientation.

S_1 We have S_1 is oriented upward (towards positive z) and parameterized by $\boldsymbol{r}(x,y) = \langle x, y, 1 - x^2 - y^2 \rangle : (x,y) \in D$ where D is the unit disc. Equation 11.4 sees

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \boldsymbol{F} \cdot \langle -g_x, -g_y, 1 \rangle$$

$$= \iint_D \langle y, x, 1 - x^2 - y^2 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA$$

$$= \iint_D 2xy + 2xy + 1 - x^2 - y^2 \,\mathrm{d}A = \iint_D 1 + 4xy - (x^2 + y^2)\,\mathrm{d}A$$

$$= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2) r \,\mathrm{d}r\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} + r^4 \cos\theta \sin\theta - \frac{1}{4} r^4\right]_0^1 \mathrm{d}\theta$$

$$= \int_0^{2\pi} \cos\theta \sin\theta - \frac{1}{4}\,\mathrm{d}\theta = \left[\frac{1}{2}\sin^2\theta - \frac{1}{4}\theta\right]_0^{2\pi} = \frac{\pi}{2}.$$

S_2 The disk S_2 has *downward orientation* and in particular $\boldsymbol{n} = \langle 0, 0, -1 \rangle$. Thus

$$\iint_{S_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}\,\mathrm{d}S = \iint_{S_2} \langle y, x, 0 \rangle \cdot \langle 0, 0, -1 \rangle \,\mathrm{d}S = \iint_{S_2} 0 \mathrm{d}S = 0.$$

FINAL ANSWER $\iint_S \langle y, x, z \rangle \cdot \boldsymbol{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$ ◆

11.5 Exercises

Question 11.10. $\iint_S y\, dS$, where S is given by

$$\boldsymbol{r}(u,v) = \langle u\cos v,\, u\sin v,\, v\rangle : (u,v) \in [0,1] \times \left[0, \frac{\pi}{2}\right].$$

ANSWER. We have $\iint_S y\, ds = \int_0^{\frac{\pi}{2}} \int_0^1 u \sin v \cdot |\boldsymbol{r}_u \times \boldsymbol{r}_v|\, dA$ where

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \left\langle \sin v,\, -\cos v,\, u\cos^2 v + u\sin^2 v \right\rangle$$

which implies $||\boldsymbol{r}_u \times \boldsymbol{r}_v|| = \sqrt{\sin^2 v + \cos^2 v + u} = \sqrt{1+u^2}$. Thereby

$$\begin{aligned}
\iint_D f(\boldsymbol{r}(x,y))\, ||\boldsymbol{r}_u \times \boldsymbol{r}_v||\, dA &= \int_0^{\frac{\pi}{2}} \int_0^1 u \sin v (1+u^2)^{\frac{1}{2}}\, du\, dv \\
&= \int_0^1 u(1+u^2)^{\frac{1}{2}}\, du \int_0^{\frac{\pi}{2}} \sin v\, dv = \frac{1}{3}\left[(1+u^2)^{\frac{3}{2}}\right]_0^1 \cdot [-\cos v]_0^{\frac{\pi}{2}} \\
&= \left(\frac{1}{3}(2)^{\frac{3}{2}} - \frac{1}{3}(1)^{\frac{3}{2}}\right)\left(-\cos\frac{\pi}{2} + \cos 0\right) = \frac{2\sqrt{2}-1}{3}.
\end{aligned}$$

◆

Question 11.11. $\iint_S x^2yz\, dS$ where S is that the plane $z = 1 + 2x + 3y$ lying above $[0,3] \times [0,2]$.

ANSWER. S is parameterized by $\boldsymbol{r}(x,\, y) = \langle x,\, y,\, 1+2x+3y\rangle$ and so

$$\boldsymbol{r}_x = \langle 1,\, 0,\, 2\rangle, \qquad \boldsymbol{r}_y = \langle 0,\, 1,\, 3\rangle, \qquad \boldsymbol{r}_x \times \boldsymbol{r}_y = \langle -2,\, -3,\, 1\rangle.$$

Thereby

$$\begin{aligned}
&\iint_D f(\boldsymbol{r}(x,y))\, ||\boldsymbol{r}_u \times \boldsymbol{r}_v||\, dA \\
&= \int_0^2 \int_0^3 x^2 y\, (1+2x+3y)(2^2+3^2+1)^{\frac{1}{2}}\, dx\, dy \\
&= \sqrt{14} \int_0^2 \int_0^3 x^2y + 2x^3y + 3x^2y^2\, dx\, dy \\
&= \sqrt{14} \int_0^2 \left[\frac{1}{3}x^3y + \frac{1}{2}x^4 + x^2y^2\right]_0^3 dy = \sqrt{14}\int_0^2 3^2y + \frac{3^4}{2} + 3^3y^2\, dy \\
&= \left[\frac{3^2}{2}y^2 + \frac{3^4}{2}y + 3^2y^3\right]_0^2 \sqrt{14} = 171\sqrt{14}.
\end{aligned}$$

◆

Question 11.12. $\iint_S (x^2z + y^2z)\,\mathrm{d}S$ where S is the hemisphere $x^2 + y^2 + z^2 = 4$ and $z \geq 0$.

ANSWER. Notice

$$\begin{aligned} S &= \{(x,y,z) : x^2 + y^2 + z^2 = 4 \wedge z \geq 0\} \\ &= \{\langle 2\sin\phi\cos\theta,\, 2\sin\phi\sin\theta,\, 2\cos\phi\rangle : (\theta,\phi) \in [0,2\pi] \times [0,\pi] = D\} \end{aligned}$$

and so S is given by $\boldsymbol{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta,\, 2\sin\phi\sin\theta,\, 2\cos\phi\rangle$ where

$$\begin{aligned} \boldsymbol{r}_\phi &= 2\left\langle \cos\phi\cos\theta,\, \cos\phi\sin\theta,\, -\sin\phi \right\rangle, \\ \boldsymbol{r}_\theta &= -2\left\langle \sin\phi\sin\theta,\, \sin\phi\cos\theta,\, 0 \right\rangle, \\ \boldsymbol{r}_\phi \times \boldsymbol{r}_\theta &= 4\left\langle \sin^2\phi\cos\theta,\, \sin^2\phi\sin\theta,\, \sin\phi\cos\phi \right\rangle. \\ ||\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta|| &= 4\sin\phi. \end{aligned}$$

Also $f(x,y,z) = x^2z + y^2z$ which implies

$$\begin{aligned} f(\boldsymbol{r}(\phi,\theta)) &= (2\sin\phi\cos\theta)^2\, 2\cos\phi + (2\sin\phi\sin\theta)^2\, 2\cos\phi \\ &= 8\sin^2\phi\cos\phi(\cos^2\theta + \sin^2\theta) = 8\sin^2\phi\cos\phi. \end{aligned}$$

Finally

$$\begin{aligned} &\iint_S (x^2z + y^2z)\,\mathrm{d}S \\ &\quad = \iint_D f(\boldsymbol{r}(\phi,\theta))\,||\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta||\,\mathrm{d}A = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 8\sin^2\phi\cos\phi\, 4\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &\quad = 32\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin^3\phi\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 32\int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \sin^3\phi\cos\phi\,\mathrm{d}\phi \\ &\quad = 32 \cdot 2\pi \cdot \left[\frac{1}{4}\sin^4\phi\right]_0^{\frac{\pi}{2}} = 16\pi\left(\sin 4\frac{\pi}{2} - \sin^4 0\right) = 16\pi(1-0) = 16\pi. \end{aligned}$$

◆

Question 11.13. Find the *flux* of $\boldsymbol{F} = \langle x, y, 5\rangle$ along the boundary of the surface enclosed by the *cylinder* $x^2 + z^2 = 1$ and the planes $y = 0$ and $y = 1$.

ANSWER. Notice in this question that y is dependent whereas x and z are independent. This mean our "up" in this coordinate system is $\langle 0, 1, 0\rangle$.

Let us say E is a cylinder with a bottom S_0 (with downward normal), lid S_1 (with upward normal), and S_2 the surface of the cylinder.

FLUX ALONG S_2. The surface of this cylinder is given by

$$S_2 = \{\boldsymbol{r}(\theta, y) = \langle \cos\theta,\, y,\, \sin\theta\rangle : (\theta, y) \in [0,1] \times [0, 2\pi] = D\}$$

and thereby

$$\boldsymbol{r}_\theta = \langle -\sin\theta,\, 0,\, \cos\theta \rangle\,, \quad \boldsymbol{r}_y = \langle 0,\, 1,\, 0 \rangle\,, \quad \boldsymbol{r}_\theta \times \boldsymbol{r}_y = \langle -\cos\theta,\, 0,\, -\sin\theta \rangle\,.$$

$$\begin{aligned}
&\iint_{S_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} \\
&\quad = \iint_D \boldsymbol{F}(\boldsymbol{r}(\theta, y)) \cdot (\boldsymbol{r}\theta \times \boldsymbol{r}_y)\, \mathrm{d}A \\
&\quad = \iint_D \langle \cos\theta,\, y,\, 5 \rangle \cdot \langle -\cos\theta,\, 0 \;\; -\sin\theta \rangle\; \mathrm{d}A \\
&\quad = \int_0^1 \int_0^{2\pi} -\cos^2\theta + 0 - 5\sin\theta\, \mathrm{d}\theta\, \mathrm{d}y = -\int_0^1 \mathrm{d}y \int_0^{2\pi} \cos^2\theta + 5\sin\theta\, \mathrm{d}\theta \\
&\quad = -\left[\frac{1}{2}\theta + \frac{1}{4}\sin^2\theta - 5\cos\theta\right]_0^{\frac{2}{\pi}} = -\left(\pi + \frac{1}{4}\sin(4\pi) - 5\cos(2\pi)\right) \\
&\quad = -(\pi + 0 - 5) = 5 - \pi.
\end{aligned}$$

FLUX ALONG S_0 For the “base” we have normal given by $\langle 0, -1, 0 \rangle$ (downward orientation). Also, the boundary of the base is the circle at $y = 0$ and so

$$S_0 = \{\boldsymbol{r}(a, \theta) = \langle a\cos\theta,\, 0,\, a\sin\theta \rangle : (a, \theta) \in [0, 1] \times [0, 2\pi] = D\}\,.$$

and thereby

$$\boldsymbol{r}_a = \langle \cos\theta,\, 0,\, \sin\theta \rangle\,, \quad \boldsymbol{r}_\theta = \langle -a\sin\theta,\, 0,\, a\cos\theta \rangle\,, \quad \boldsymbol{r}_a \times \boldsymbol{r}_\theta = \langle 0,\, -a,\, 0 \rangle\,.$$

Notice $\boldsymbol{r}_{\boldsymbol{a}} \times \boldsymbol{r}_\theta$ has the correct orientation. Finally

$$\int_{S_0} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint_D \langle a\cos\theta,\, 0,\, 5 \rangle \cdot \langle 0,\, -a,\, 0 \rangle\; \mathrm{d}a\, \mathrm{d}\theta = \int_0^1 \int_0^{2\pi} 0\, \mathrm{d}a\, \mathrm{d}\theta = 0.$$

FLUX ALONG S_1 For the “lid” we have normal given by $\langle 0, 1, 0 \rangle$ (upward orientation). Also, the boundary of the lid is the circle at $y = 1$ and so

$$S_1 = \{\boldsymbol{r}(a, \theta) = \langle a\cos\theta,\, 1,\, a\sin\theta \rangle : (a, \theta) \in [0, 1] \times [0, 2\pi] = D\}$$

and thereby

$$\boldsymbol{r}_a = \langle \sin\theta,\, 0,\, \cos\theta \rangle\,, \quad \boldsymbol{r}_\theta = \langle a\cos\theta,\, 0,\, \cos\theta \rangle\,, \quad \boldsymbol{r}_a \times \boldsymbol{r}_b = \langle 0,\, a,\, 0 \rangle\,.$$

Notice $\boldsymbol{r}_a \times \boldsymbol{r}\theta$ has the correct orientation. Finally

$$\iint_{S_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \int_0^{2\pi} \langle a\sin\theta, 1, 5 \rangle \cdot \langle 0, a, 0 \rangle\; \mathrm{d}\theta \mathrm{d}a = \int_0^1 \int_0^{2\pi} a\; \mathrm{d}\theta \mathrm{d}a$$

$$= \int_0^1 a \, \mathrm{d}a \int_0^{2\pi} \mathrm{d}\theta = \left[\frac{a^2}{2}\right]_0^1 \cdot 2\pi = \pi.$$

FINAL ANSWER $\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = 0 + (5 - \pi) + \pi = 5.$ ◆

12 Stokes and Divergence Theorems

Stoke's Theorem is the higher-dimensional version of Green's Theorem. *Green's Theorem* relates a *double integral* over a planar region D to the *line integral* around ∂D (a planar curve). *Stoke's Theorem* relates a *surface integral* over the surface S to a *line integral* along ∂S (a space curve).

12.1 Stokes Theorem

Stoke's Theorem. Let S be an *oriented*, *piecewise-smooth* surface bounded by a *simple*, *closed*, *piecewise-smooth* boundary ∂S with positive orientation. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field with continuous partial derivatives. Then

$$\oint_{\partial S} \boldsymbol{F} \cdot \mathrm{d}r = \iint_S \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}S.$$

PROOF. See textbook for special case where S is given by a function. ■

Notice when S is planar in xy with unit normal $\langle 0, 0, 1 \rangle$ then *Stokes*:

$$\oint_{\partial S} \boldsymbol{F} \cdot \mathrm{d}r = \iint_S \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}S.$$

implies Green's Theorem

$$\oint_{\partial S} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_S \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot \boldsymbol{k}\, \mathrm{d}A.$$

Question 12.1. Evaluate $\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ where $\boldsymbol{F}(x, y, z) = \langle -y^2, x, z^2 \rangle$ and C is the intersection of $y + z = 2$ and $x^2 + y^2 = 1$. That is,

$$C = \mathcal{G}(y + z - 2) \cap \mathcal{G}(x^2 + y^2 - 1).$$

Note: We could evaluate $\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ directly if we wanted.

ANSWER. By Stokes' Theorem $\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_S \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$.

$$\operatorname{curl} \boldsymbol{F} = \operatorname{curl} \left\langle -y^2, x, z^2 \right\rangle = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0-0, 0-0, 1+2y \rangle .$$

Although there are many S such that $C = \partial S$ the most convenient is the elliptical region given by $y + z = 2$ *above the unit disc* $D_{r\theta} = [0,1] \times [0, 2\pi)$.

$$\begin{aligned} \oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &= \iint_S \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint_D \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}A \\ &= \iint_D \langle 0,0,1+2y \rangle \cdot \langle 0,0,1 \rangle \ \mathrm{d}A = \iint_D 1 + 2y \, \mathrm{d}A \\ &= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta) \, r \ \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi} \left[\frac{1}{2} r^2 + r^2 \sin\theta \right]_0^1 \ \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{2} + \sin\theta \ \mathrm{d}\theta \\ &= \left[\frac{1}{2}\theta - \cos\theta \right]_0^{2\pi} = (\pi - 1) - (0 - 1) = \pi. \end{aligned}$$

◆

Stoke's Theorem is used in the other direction.

Question 12.2. Compute $\iint_S \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$ where $\boldsymbol{F} = \langle xz, yz, xy \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ lying *inside* the cylinder $x^2 + y^2 = 1$ *above* the xy plane.

ANSWER. The region S is given by

$$S = \mathcal{G}(x^2 + y^2 + z^2 = 4) \cap \mathcal{G}(x^2 + y^2 \leq 1) \cap \mathcal{G}(z > 0).$$

Notice $(x^2 + y^2 + z^2 = 4) \wedge (x^2 + y^2 = 1) \wedge (z > 0) \implies (1 + z^2 = 4) \wedge (z > 0) \implies (z = \sqrt{3})$. Thus

$$\partial S = (\text{unit circle at } z = \sqrt{3}) = \left\langle \cos\theta, \sin\theta, \sqrt{3} \right\rangle = \boldsymbol{r}(\theta)$$

$$\begin{aligned} \iint_S \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} &= \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) \ \mathrm{d}t \\ &= \int_0^{2\pi} \left\langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, \cos\theta\sin\theta \right\rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \ \mathrm{d}t \\ &= \int_0^{2\pi} (-\sqrt{3}\cos\theta\sin\theta + \sqrt{3}\sin\theta\cos\theta) \ \mathrm{d}t = \sqrt{3} \int_0^{2\pi} 0 \ \mathrm{d}t = 0 \end{aligned}$$

◆

Notice here we computed a *surface integral* by knowing the values of $\boldsymbol{F}$ on a boundary curve C. This means *any* other oriented surface with the same boundary yields the same surface integral!

Proposition 12.3. Suppose $\partial S_1 = C = \partial S_2$. Then

$$\iint_{S_1} \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \iint_{S_2} \operatorname{curl} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

Use this proposition when it is difficult to integrate over one surface but easy over the other.

12.2 *The Divergence Theorem*

Green's Theorem in its vector form is

$$\oint_{\partial D} \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}s = \iint_D \operatorname{div} \boldsymbol{F}(x, y) \, \mathrm{d}A$$

where D is a simple *region* and ∂D is *positively oriented* We wish to extend this to vector fields on $\mathbb{R}^3$ like (perhaps)

$$\iint_{\partial E} \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S = \iiint_E \operatorname{div} \boldsymbol{F}(x, y, z) \, \mathrm{d}V$$

where E is a simple *solid*. Indeed under the appropriate hypothesis this is the correct extension.

The Divergence Theorem. Let E be a *simple solid region* and let ∂E have positive (outward) orientation. Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field whose components have continuous partial derivatives. Then

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iiint_E \operatorname{div} \boldsymbol{F} \, \mathrm{d}V.$$

PROOF. Omitted. See textbook. ■

Question 12.4. Find the flux of $\boldsymbol{F} = \langle z, y, x \rangle$ over the unit sphere.

ANSWER. The unit sphere bounds the unit ball given by $B_{\phi\theta r} = [0, \pi] \times [0, 2\pi) \times [0, 1]$ and also we have $\operatorname{div} \boldsymbol{F} = \nabla \cdot \langle z, y, x \rangle = 0 + 1 + 0$ so

$$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iiint_B \operatorname{div} \boldsymbol{F} \, \mathrm{d}V = \iiint_B 1 \, \mathrm{d}V$$

Convert to spherical

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^1 r^2 \sin\phi \, dr \, d\theta \, d\phi = \int_0^{2\pi} d\theta \int_0^1 r^2 \, dr \int_0^{\pi} \sin\phi \, d\phi$$

$$= [\theta]_0^{2\pi} \cdot \left[\frac{r^3}{3}\right]_0^1 \cdot [-\cos\phi]_0^{\pi} = (2\pi) \cdot \left(\frac{1}{3}\right) \cdot (-\cos\pi + \cos 0) = \frac{4}{3}\pi.$$

◆

Question 12.5. Evaluate $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ where $F = \left\langle xy, y^2 + e^(xz^2), \sin(xy) \right\rangle$ and S is the surface of the region E bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, $y + z = 2$.

ANSWER. In order to determine the integral directly we would have to evaluate *four* surface integrals corresponding to the four pieces of S. Moreover, the div $\boldsymbol{F}$ is fairly simply:

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \left\langle xy, y^2 + e^(xz^2), \sin(xy) \right\rangle = (y) + (2y + 0) + (0) = 3y.$$

We use Divergence Theorem to transform $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ into $\iiint_E \operatorname{div} \boldsymbol{F} \, dV$.

Recall E is bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, $y + z = 2$ so

$$E_{xzy} = [-1, 1] \times [0, 1 - x^2] \times [0, 2 - z]$$

and thus

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \operatorname{div} \boldsymbol{F} \, dV = \iint_E 3y \, dV$$

$$= \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} 3y \, dy \, dz \, dx = \int_{-1}^1 \int_0^{1-x^2} \left[\frac{3}{2}y^2\right]_0^{2-z} dz \, dx$$

$$= \frac{3}{2} \int_{-1}^1 \int_0^{1-x^2} (2-z)^2 \, dz \, dx = \frac{3}{2} \int_{-1}^1 \left[-\frac{1}{3}(2-z)^3\right]_0^{1-x^2} dx$$

$$= -\frac{3}{2}\frac{1}{3} \int_{-1}^1 \left[(2 - 1 + x^2)^3 - 2^3\right]_0^{1-x^2} dx = -\frac{1}{2} \int_{-1}^1 (1 + x^2)^3 - 8 \, dx$$

$$= -\frac{1}{2} \int_{-1}^1 x^6 + 3x^4 + 3x^2 - 7 \, dx = -\frac{1}{2} \left[\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 - 7x\right]_{-1}^1$$

$$= -\frac{1}{2} \left(\left(\frac{1}{7} + \frac{3}{5} + 1 - 7\right) - \left(-\frac{1}{7} - \frac{3}{5} - 1 + 7\right) \right)$$

$$= -\frac{1}{2} \left(-\frac{185}{35} - \frac{185}{35}\right) = \frac{185}{35}.$$

◆

Divergence Theorem also works for finite unions of simple solid regions.

Proposition 12.6. Let E be a simple solid and let $\partial E = S = S_1 \cup S_2$ then

$$\iiint_E \operatorname{div} \boldsymbol{F} \, \mathrm{d}V = -\iint_{S_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} + \iint_{S_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}.$$

PROOF. $\iint_E \operatorname{div} \boldsymbol{F} \, \mathrm{d}V = \iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S = \iint_{S_1} \boldsymbol{F} \cdot (-\boldsymbol{n}_1) \, \mathrm{d}S + \iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}_2 \, \mathrm{d}S = -\iint_{S_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} + \iint_{S_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$. ■

12.3 Exercises

12.3.1 Stoke's Theorem

Question 12.7. Evaluate $\iint_S \operatorname{curl} \boldsymbol{F} \cdot d\boldsymbol{S}$. $\boldsymbol{F} = \langle xyz,\ xy,\ x^2yz \rangle$, S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1,\ \pm 1,\ \pm 1)$, oriented outward.

ANSWER. The boundary of S is given by the line segments $(1, -1, -1)$ to $(1, 1, -1)$; $(1, 1-1)$ to $(-1, 1, -1)$; $(-1, 1-1)$ to $(-1, -1, -1)$; $(-1, -1, -1)$ to $(1, -1, -1)$. These correspond to the curves / parametric lines

C_0: $\boldsymbol{r}_0 = \langle 1,\ -1,\ -1 \rangle + t\,\langle 1-1,\ 1+1,\ -1+1 \rangle = \langle 1,\ 2t-1,\ -1 \rangle$,

C_1: $\boldsymbol{r}_1 = \langle 1,\ 1,\ -1 \rangle + t\,\langle -1-1,\ 1-1,\ -1+1 \rangle = \langle 1-2t,\ 1,\ -1 \rangle$,

C_2: $\boldsymbol{r}_2 = \langle -1,\ -1,\ -1 \rangle + t\,\langle 1+1,\ -1+1,\ -1+1 \rangle = \langle -1,\ 1-2t,\ -1 \rangle$, and

C_3: $\boldsymbol{r}_3 = \langle -1,\ -1,\ -1 \rangle + t\,\langle 1+1,\ -1+1,\ -1+1 \rangle = \langle 2t-1,\ -1,\ -1 \rangle$

where $t \in [0, 1]$ for everything above.

Since $\iint_S \operatorname{curl} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_{C_1 \cup \cdots \cup C_3} \boldsymbol{F} \cdot d\boldsymbol{r}$ the answer is the sum on the following four integrals.

$$\int_{C_0} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_t \boldsymbol{F}(\boldsymbol{r}_0(t)) \cdot \boldsymbol{r}_0'(t)\ dt = \int_t \langle \_,\ (1)(2t-1),\ \_ \rangle \cdot \langle 0, 2, 0 \rangle\ dt$$
$$= \int_0^1 4t - 2\ dt = \left[2t^2 - 2t\right]_0^1 = 0$$

Similarly

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r}_1 = \int_0^1 2t - 1\ dt = 0$$
$$\int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}_2 = \int_0^1 2t - 1\ dt = 0$$
$$\int_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r}_3 = \int_0^1 2t - 1\ dt = 0$$

and thus $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = 0$ by Stoke's Theorem. ◆

Question 12.8. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$. $\boldsymbol{F} = \langle x + y^2,\ y + z^2,\ z + x^2 \rangle$, C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

ANSWER. If $C = \partial S$ then S is the *plane* through (say) $(1,0,0)$ given by the normal of (say) the two vectors connecting $(1,0,0)$ and $(0,1,0)$ to $(0,0,1)$.

$$\begin{aligned}&\langle x-1,\, y,\, z\rangle \cdot \langle 0-1,\, 0-0,\, 1-0\rangle \times \langle 0-1,\, 1-0,\, 0-0\rangle\\ &= \langle x-1,\, y,\, z\rangle \cdot \langle 0-1,\, 1,\, -1\rangle = 1-x+y-z\end{aligned}$$

This means $z = 1-x+y$ and thereby

$$S = \{\boldsymbol{r}(x,y) = (x,\, y\,,\, 1-x+y) : (x,y) \in [0,\, 1] \times [0,\, 1-x] = D\}\,.$$

By Stoke's Theorem we have

$$\begin{aligned}\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} &= \iint_D \operatorname{curl} \boldsymbol{F}(\boldsymbol{r}(x,y)) \cdot \boldsymbol{r}_x \times \boldsymbol{r}_y \,\mathrm{d}A\\ &= \iint_D \langle -2z,\, 2x,\, -2y\rangle_{(x,y,z)=(x,y,1-x+y)} \cdot \langle 1,0,-1\rangle \times \langle 0,\, 1,\, 1\rangle \,\mathrm{d}A\\ &= \iint_D \langle -2(1-x+y),\, 2x,\, -2y\rangle \cdot \langle 1,\, 1,\, 1\rangle \,\mathrm{d}A\\ &= \iint_D 2x-2y-2+2x-2y \,\mathrm{d}A = \int_0^1 \int_0^{1-x} 4x-4y-2 \,\mathrm{d}y\mathrm{d}x\\ &= \int_0 1\, \left[4xy-2y^2-2y\right]_0^{1-x} \,\mathrm{d}x = \int_0^1 -6x^2+10x-4 \,\mathrm{d}x\\ &= \left[-2x^3+5x^3-4x\right]_0^1 = -2+5-4 = -1.\end{aligned}$$

Thus $\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = -1$. ◆

12.3.2 Divergence

Question 12.9. Verify the Divergence Theorem is true for the vector field $\boldsymbol{F} = \langle 3x,\, xy,\, 2xz\rangle$ on the region E: the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

BY STOKE'S THEOREM. We have the *solid* E is given by

$$E_{xyz} = [0,1] \times [0,1] \times [0,1]$$

and $\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = 3+x+2x$. Thereby

$$\begin{aligned}\iint_{\partial E} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{S} &= \iiint_E \operatorname{div} \boldsymbol{F} \,\mathrm{d}A\\ &= \int_0^1 \int_0^1 \int_0^1 3x+3 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 3\left[\frac{x^2}{2}+x\right]_0^1 \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z = \frac{9}{2}\end{aligned}$$

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BY DIVERGENCE THEOREM. The cube has *six* faces

Surface	Located	Normal
S_0	$x = 0$	$\langle -1, 0, 0 \rangle$
S_1	$x = 1$	$\langle 1, 0, 0 \rangle$
S_2	$y = 0$	$\langle 0, -1, 0 \rangle$
S_3	$y = 1$	$\langle 0, 1, 0 \rangle$
S_4	$z = 0$	$\langle 0, 0, -1 \rangle$
S_5	$z = 1$	$\langle 0, 0, 1 \rangle$.

and we have by $\iint_{\partial E} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint_{S_0} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} + \cdots + \iint_{S_5} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$. Zero contributions to the integral are made by

$$\iint_{S_0} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \int_0^1 \langle 0,\ 0,\ 0 \rangle \cdot \langle -1,\ 0,\ 0 \rangle \ \mathrm{d}x\,\mathrm{d}y = 0$$
$$\iint_{S_3} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \int_0^1 \langle 3x,\ 0,\ 2xz \rangle \cdot \langle 0,\ -1,\ 0 \rangle \ \mathrm{d}x\,\mathrm{d}y = 0$$
$$\iint_{S_4} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \int_0^1 \langle 3x,\ xy,\ 0 \rangle \cdot \langle 0,\ 0,\ -1 \rangle \ \mathrm{d}x\,\mathrm{d}y = 0$$

whereas

$$\int_{S_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \int_0^1 \langle 3,\ y,\ 2z \rangle \cdot \langle 1,\ 0,\ 0 \rangle \ \mathrm{d}x\,\mathrm{d}y = 3,$$
$$\int_{S_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \int_0^1 \langle 3x,\ x,\ 2xz \rangle \cdot \langle 0,\ 1,\ 0 \rangle \ \mathrm{d}x\,\mathrm{d}y = \frac{1}{2},$$
$$\int_{S_5} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \int_0^1 \int_0^1 \langle 3x,\ xy,\ 2x \rangle \cdot \langle 0,\ 0,\ 1 \rangle \ \mathrm{d}x\,\mathrm{d}y = 1.$$

And thus the integral is given by $3 + \frac{1}{2} + 1 = \frac{9}{2}$ which matches the Stoke's Theorem calculation. ◆

Question 12.10. Calculate the flux $\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$. $\boldsymbol{F} = \langle 3xy^2,\ xe^z,\ z^3 \rangle$, S is the surface bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

ANSWER. If $\partial E = S$ then the *solid* E is (in cylindrical) given by $E_{r\theta x} = [0, 1] \times [0, 2\pi] \times [-1, 2]$ and so

$$\begin{aligned}\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} &= \iiint_E \operatorname{div} \boldsymbol{F} \,\mathrm{d}V \\ &= \iiint_E 3y^2 + 3z^2 \,\mathrm{d}V = 3 \int_0^1 \int_0^{2\pi} \int_{-1}^2 (r^2 \cos^2\theta + r^2 \sin^2\theta)\, r \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}x \\ &= 3 \int_0^1 \int_0^{2\pi} \int_{-1}^2 r^3 \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}x = \frac{3}{4}(2+1)(2\pi - 0) \left[r^4 \right]_0^1 = \frac{9}{2}\pi.\end{aligned}$$

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