ORCCA
Ontario Research Centre for Computer Algebra

— On Fulton's Algorithm for Computing Intersection Multiplicities —

Steffen Marcus, Marc Moreno Maza, and Paul Vrbik

Western

Introduction

Let $f_1, \ldots, f_n \in \mathbf{k}[x_1, \ldots, x_n]$ such that $\mathbf{V}(f_1, \ldots, f_n) \subset \overline{\mathbf{k}}[x_1, \ldots, x_n]$ is zero-dimensional. The intersection multiplicity $I(p; f_1, \ldots, f_n)$ at the point $p \in \mathbf{V}(f_1, \ldots, f_n)$ specifies the *weights* of the weighted sum in Bézout's Theorem.

The number $I(p; f_1, \ldots, f_n)$ is not natively computable by MAPLE while it is computable by SINGULAR and MAGMA—but only when all coordinates of p are in k.

We are interested in removing this algorithmic limitation. We combine Fulton's Algorithm and the theory of regular chains, leading to a complete algorithm for $n = 2$. Moreover, we propose algorithmic criteria for reducing the case of $n > 2$ variables to the bivariate one. Experimental results are reported.

The case of two plane curves

Intuitively, the intersection multiplicity (IM) of two plane curves at a given point counts the number of times that these curves intersect at that point. More formally, given an arbitrary field $\mathbf{k}$ and two bivariate polynomials $f, g \in \mathbf{k}[x, y]$, consider the affine algebraic curves $\mathcal{C} := \mathbf{V}(f)$ and $\mathcal{D} := \mathbf{V}(g)$ in $\mathbb{A}^2 = \overline{\mathbf{k}}^2$, where $\overline{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$. Let p be a point in the intersection.

The intersection multiplicity of p in $\mathbf{V}(f, g)$ is defined to be

$$I(p; f, g) = \dim_{\overline{\mathbf{k}}}(\mathcal{O}_{\mathbb{A}^2,p} / \langle f, g \rangle)$$

where $\mathcal{O}_{\mathbb{A}^2,p}$ and $\dim_{\overline{\mathbf{k}}}(\mathcal{O}_{\mathbb{A}^2,p} / \langle f, g \rangle)$ are the local ring at p and the dimension of the vector space $\mathcal{O}_{\mathbb{A}^2,p} / \langle f, g \rangle$.

Remarkably, and as pointed out by Fulton in his *Intersection Theory*, the intersection multiplicities of the plane curves C and D satisfy a series of properties which uniquely define $I(p; f, g)$ at each point $p \in \mathbf{V}(f, g)$. Moreover, the proof of this remarkable fact is constructive, which leads to an algorithm.

▷ Fulton's Properties ◁

The intersection multiplicities of two plane curves satisfy and are uniquely determined by the following.

1. $I(p; f, g)$ is a non-negative integer for any C, D, and p such that C and D have no common component at p. We set $I(p; f, g) = \infty$ if C and D have a common component at p.
2. $I(p; f, g) = 0$ if and only if $p \notin C \cap D$.
3. $I(p; f, g)$ is invariant under affine change of coordinates on $\mathbb{A}^2$.
4. $I(p; f, g) = I(p; g, f)$
5. $I(p; f, g)$ is greater or equal to the product of the multiplicity of p in f and g, with equality occurring if and only if C and D have no tangent lines in common at p.
6. $I(p; f, gh) = I(p; f, g) + I(p; f, h)$ for all $h \in \mathbf{k}[x, y]$.
7. $I(p; f, g) = I(p; f, g + hf)$ for all $h \in \mathbf{k}[x, y]$.

Fulton's Algorithm — $\mathsf{IM}_2(p; f_1, f_2)$

Input: $p = (\alpha, \beta) \in \mathbb{A}^2(\mathbf{k})$ and $f, g \in \mathbf{k}[y \succ x]$ such that $\mathbf{gcd}(f, g) \in \mathbf{k}$
Output: $I(p; f, g) \in \mathbb{N}$ satisfying (2-1)–(2-7)

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if f(p) ≠ 0 or g(p) ≠ 0 then
    return 0;
r, s = deg(f(x, β)), deg(g(x, β)); assume s ≥ r.
if r = 0 then
    write f = (y − β) · h and g(x, β) = (x − α)^m (a_0 + a_1(x − α) + ···);
    return m + IM_2(p; h, g);
    IM_2(p; (y − β) · h ∩ g) = IM_2(p; (y − β), g) + IM_2(p; h, g)
    IM_2(p; (y − β) ∩ g) = IM_2(p; (y − β) ∩ g(x, β)) = IM_2(p; (y − β) ∩ (x − α)^m) = m
if r > 0 then
    h ← monic(g) − (x − α)^(s−r) monic(f);
    return IM_2(p; f, h);
```

Our Goal: Extending Fulton's Algorithm

Limitations of Fulton's Algorithm:

- does not generalize to $n > 2$, that is, to n polynomials $f_1, \ldots, f_n \in \mathbf{k}[x_1, \ldots, x_n]$ since $\mathbf{k}[x_1, \ldots, x_{n-1}]$ is no longer a PID.
- is limited to computing the IM at a single point with rational coordinates, that is, with coordinates in the base field $\mathbf{k}$. (Approaches based on standard or Gröbner bases suffer from the same limitation)

▷ Our contributions ◁

- We adapt Fulton's Algorithm such that it can work at any point of $\mathbf{V}(f_1, f_2)$, rational or not.
- For $n \geq 2$, we propose an algorithmic criterion to reduce the n-variate case to that of $n-1$ variables.

▷ Our tools ◁

Regular Chains

To deal with non-rational points, we extend Fulton's Algorithm to compute $\mathsf{IM}_2(T; f_1, x_2)$, where $T \subset \mathbf{k}[x_1, x_2]$ is a regular chain such that we have $\mathbf{V}(T) \subseteq \mathbf{V}(f_1, f_2)$. This makes sense thanks to the following theorem.

Theorem 1. Recall that $\mathbf{V}(f_1, f_2)$ is zero-dimensional. Let $T \subset \mathbf{k}[x_1, x_2]$ be a regular chain such that we have $\mathbf{V}(T) \subset \mathbf{V}(f_1, f_2)$ and the ideal $\langle T \rangle$ is maximal. Then $\mathsf{IM}_2(p; f_1, f_2)$ is the same at any point $p \in \mathbf{V}(T)$.

Expansions About a Set of Points

We observe that this algorithm works the Taylor series of f_1, f_2 at a rational point p. To extend this idea when working with $\mathbf{V}(T)$, instead of a point p, we introduce two new variables y_1 and y_2 representing $x_1 - \alpha$ and $x_2 - \beta$ respectively, for an arbitrary point $(\alpha, \beta) \in \mathbf{V}(T)$. These variables are simply used as place holders in the following definition, where $f \in \{f_1, f_2\}$.

Let $F \in \mathbf{k}[x_1, x_2][y_1, y_2]$ and $T \subset \mathbf{k}[x_1, x_2]$ be a regular chain such that we have $\mathbf{V}(T) \subset \mathbf{V}(f_1, f_2)$. We say that F is an expansion of f about $\mathbf{V}(T)$ if at every point $(\alpha, \beta) \in \mathbf{V}(T)$ we have $F(\alpha, \beta)(x_1 - \alpha, x_2 - \beta) = f(x_1, x_2)$. The fundamental example is $F = \sum_j \left(\sum_i f_{i,j} y_1^i\right) y_2^j$ where $f_{i,j} = \frac{1}{i!j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}$.

▷ Our algorithm for the bivariate case ◁

For an arbitrary zero-dimensional regular chain T, we apply the D5 Principle to Fulton's Algorithm in order to reduce to the irreducible case, as covered by the previous theorem.

Algorithm — $\mathsf{IM}_2(T; F^1, F^2)$

Input: F^1 and F^2 as described in the previous slide.
Output: Finitely many pairs (T_i, m_i) where $T_i \subset k[x_1, x_2]$ are regular chains and $m_i \in \mathbb{Z}^+$ such that $\forall p \in V(T^i)$ $I(p; f_1, f_2) = m_i$.

```
for (F_1^1, T) ∈ Regularize(F_1^1, T) do
    if F_1^1 ∉ ⟨T⟩ then
        output(T, 0);
    else
        for (T, F_1^2) ∈ Regularize(F_1^2, T) do
            if F_1^2 ∉ ⟨T⟩ then
                output(T, 0);
            else
                for (T, a_{F^1}) ∈ LT(F^1_{<y_2}, T) do
                    for (T, a_{F^2}) ∈ LT(F^2_{<y_2}, T) do
                        /* Wlog deg(F^1_{<y_2}) ≤ deg(F^2_{<y_2}) */
                        if a_{F^1} ∈ ⟨T⟩ then
                            for (T, d) ∈ TDeg(F^2_{<y_2}, T) do
                                for (T, i) ∈ IM_2(T, (F^1 − F^1_{<y_2})/y_2, F^2) do
                                    output(T, (d + i));
                        else
                            H ← F^2 − a_{F^2} · Inverse(a^1_F, T) · F^1;
                            output(IM_2(T, F^1, H));
```

Notations

In the adjacent algorithm, the polynomials F_1^1 and F_1^2 consist of the terms of F^1 and F^2 of degree 0 in both y_1 and y_2. The command Regularize $\left(F_1^1, T\right)$ separates the points of $\mathbf{V}(T)$ cancelling F_1^1 from the others. The command LT $\left(F^1_{<y_2}, T\right)$ partitions $\mathbf{V}(T)$ according to the degree of $F^1_{<y_2}$, thus computing the leading term of $F^1_{<y_2}$ at each point of $\mathbf{V}(T)$. The command TDeg $\left(F^2_{<y_2}, T\right)$ works similarly but deals with the trailing degree instead.

Reducing the n-dimensional case to the $n-1$ case

The intersection multiplicity of p in $\mathbf{V}(f_1, \ldots, f_n)$ is given by

$$I(p; f_1, \ldots, f_n) := \dim_{\overline{k}}\left(\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle\right).$$

where $\mathcal{O}_{\mathbb{A}^n,p}$ and $\dim_{\overline{k}}(\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle)$ are respectively the local ring at the point p and the dimension of the vector space $\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle$. The next theorem reduces the n-dimensional case to $n-1$, under assumptions which state that f_n does not contribute to $I(p; f_1, \ldots, f_n)$.

Theorem 2. Assume that $h_n = \mathbf{V}(f_n)$ is non-singular at p. Let v_n be its tangent hyperplane at p. Assume that h_n meets each component (through p) of the curve $\mathcal{C} = \mathbf{V}(f_1, \ldots, f_{n-1})$ transversely (that is, the tangent cone $TC_p(\mathcal{C})$ intersects v_n only at the point p). Let $h \in \mathbf{k}[x_1, \ldots, x_n]$ be the degree 1 polynomial defining v_n. Then, we have $I(p; f_1, \ldots, f_n) = I(p; f_1, \ldots, f_{n-1}, h)$.

The reduction in practice

How to use this theorem in practice? Assume that the coefficient of x_n in h is non-zero, thus $h = x_n - h'$, where $h' \in \mathbf{k}[x_1, \ldots, x_{n-1}]$. Hence, we can rewrite the ideal $\langle f_1, \ldots, f_{n-1}, h \rangle$ as $\langle g_1, \ldots, g_{n-1}, h \rangle$ where g_i is obtained from f_i by substituting x_n with h'. If instead of a point p, we have a zero-dimensional regular chain $T \subset \mathbf{k}[x_1, \ldots, x_n]$, we use the techniques developed before.

When this reduction does not apply a priori, one can look for a more favorable system of generators. For instance, consider the system *Ojika 2*:

$$x^2 + y + z - 1 = x + y^2 + z - 1 = x + y + z^2 - 1 = 0. \quad (1)$$

The above theorem does not apply. However, if one uses the first equation, say $x^2 + y + z - 1 = 0$, to eliminate z from the other two, we obtain two bivariate polynomials $f, g \in \mathbf{k}[x, y]$. At any point of $p \in \mathbf{V}(h, f, g)$ the tangent cone of the curve $\mathbf{V}(f, g)$ is independent of z; in some sense it is "vertical". On the other hand, at any point of $p \in \mathbf{V}(h, f, g)$ the tangent space of $\mathbf{V}(h)$ is not vertical. Thus, the previous theorem applies without computing any tangent cones.

Experimental Results

Label	Name	terms	degree
1	hard_one	30	37
2	L6_circles	4	24
3	spiral29_24	63	52
4	tryme	38	59
5	challenge_12	49	30
6	challenge_12_1	64	40
7	compact_surf	52	18
8	degree_6_surf	467	42
9	mignotte_xy	81	64
10	SA_4_4_eps	63	33
11	spider	292	36

System	Dim	Time(△ize)	#rc's	Time(rc_im)
$\langle 1, 3\rangle$	888	9.7	20	19.2
$\langle 1, 4\rangle$	1456	226.0	8	9.023
$\langle 3, 5\rangle$	1413	22.5	27	28.6
$\langle 4, 5\rangle$	1781	218.4	9	13.9
$\langle 5, 1\rangle$	1759	113.0	10	15.8
$\langle 6, 9\rangle$	2560	299.3	10	22.9
$\langle 6, 11\rangle$	1440	59.8	17	27.5
$\langle 7, 8\rangle$	1152	32.8	12	16.2
$\langle 7, 9\rangle$	756	18.5	16	11.2
$\langle 7, 11\rangle$	648	9.2	25	11.1
$\langle 8, 10\rangle$	1362	232.5	7	9.3
$\langle 8, 11\rangle$	1256	49.6	17	45.7
$\langle 9, 10\rangle$	2080	504.9	12	34.812
$\langle 10, 11\rangle$	1180	40.9	17	21.3

Name	Dim	Points	△ize	Cones	COV	rc_im	Total	Success
Nbody5	99	49	1.60	0.00	0.06	1.90	2.00	51/99
mth191	27	18	0.56	5400.00	0.04	0.01	5400.00	23/27
ojika2	8	5	0.20	8.20	0.13	0.47	8.80	8/8
E-Arnold1	45	30	0.89	1100.00	0.01	1800.00	2900.00	45/45
ShiftedCubes	27	25	0.66	0.00	0.00	0.52	0.52	27/27