— Visualization of Homotopy's and their Properties —

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What Is Homotopy Continuation?

Homotopy continuation, like Newton's method, is an iterative approach for finding the isolated complex roots of a polynomial. But unlike Newton's method this process guarantees every isolated root will be found if seeded by a finite number of appropriately chosen starting points. The underlying idea is:

1. *Connect* the solutions of an exactly solvable system, $q(z) = 0$ with the solutions of the desired (target) system, $p(z) = 0$:

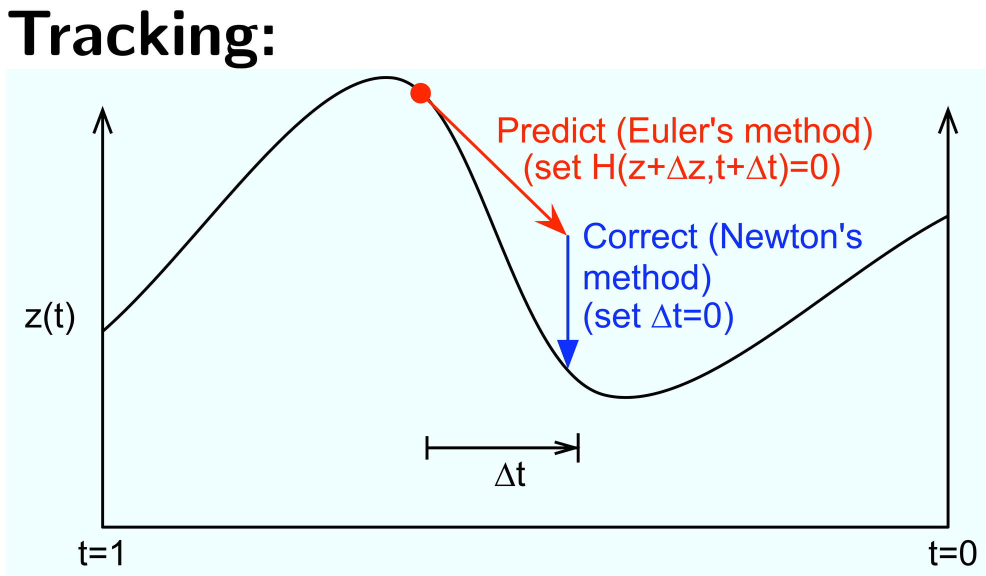
$$H(z,t) = tq(z) + (1-t)p(z).$$

$H(z,1) = q(z)$ is the start system, e.g. $z^d - 1$.
$H(z,0) = p(z)$ is the target system.

2. *Track* the solution path as t goes from 1 to 0, using:

$$\frac{dH}{dt} = H_z \frac{dz}{dt} + H_t = 0.$$

Tracking:



For basic prediction and correction, consider the first order taylor approximation.

$$H(z+\Delta z, t+\Delta t) = H(z,t) + H_z(z,t)\Delta z + H_t(z,t)\Delta t$$

To the left; given (z_1, t_1) such that $H(z_1, t_1) \approx 0$ one can predict a new approximate solution at $t_1 + \Delta t$.

Path Tracking In Action

The bad news:

Intersecting paths A tracker may actually *jump paths* and converge to the wrong root.

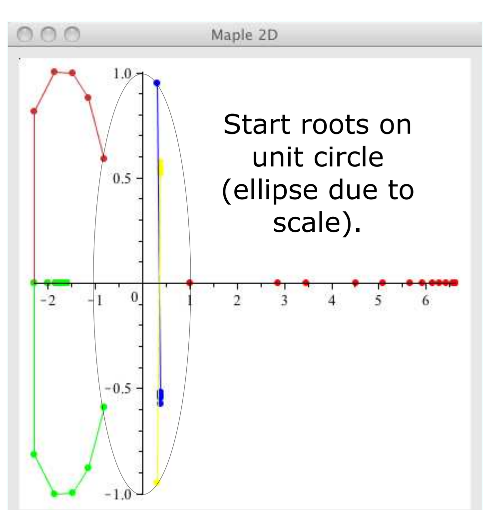
Multiple roots Newton's method converges slowly to such roots.

Start systems Choosing one is non-trivial because we can not tell exactly how many roots the target system has. This results in wasted computation.

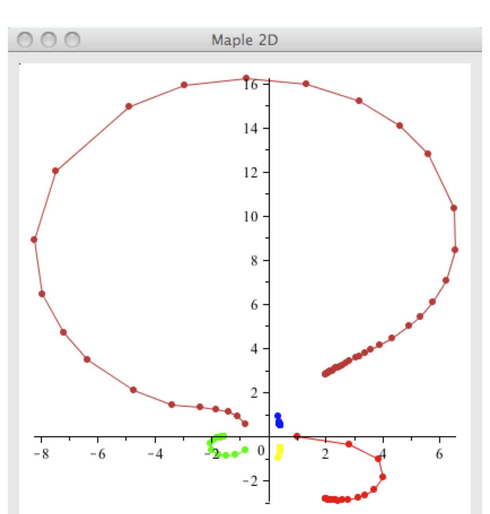
An example where tracking fails.

$$-7x^5 + 22x^4 - 55x^3 - 94x^2 + 87x - 56. \qquad (1)$$

The brown and green paths converge to the real root -1.6 whereas the blue and yellow paths converge to the complex root 0.4-0.5i. The red root is escaping to infinity and is (quickly) flagged as a failed path. We do not find all roots.



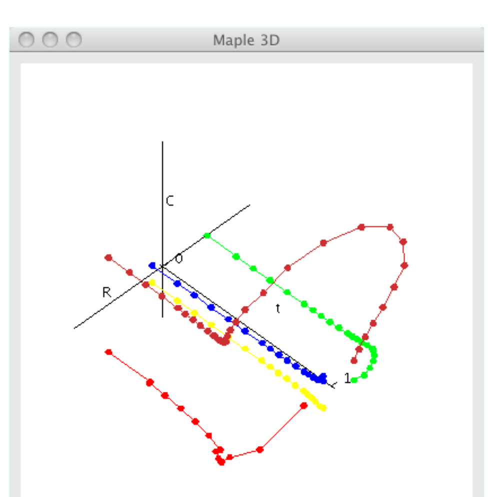
One can guarantee (with probability 1) that we find *all* roots.



Let's introduce the random components, $\theta, \phi \in [-\pi, \pi]$ and modify our original homotopy to

$$q(z) = te^{i\theta}(z^d - e^{i\phi}).$$

Now we see that the paths are (more or less) well behaved. Note, in the diagrams left and right the paths start from the unit circle. For more information on this "gamma trick" (including justification), see [1].



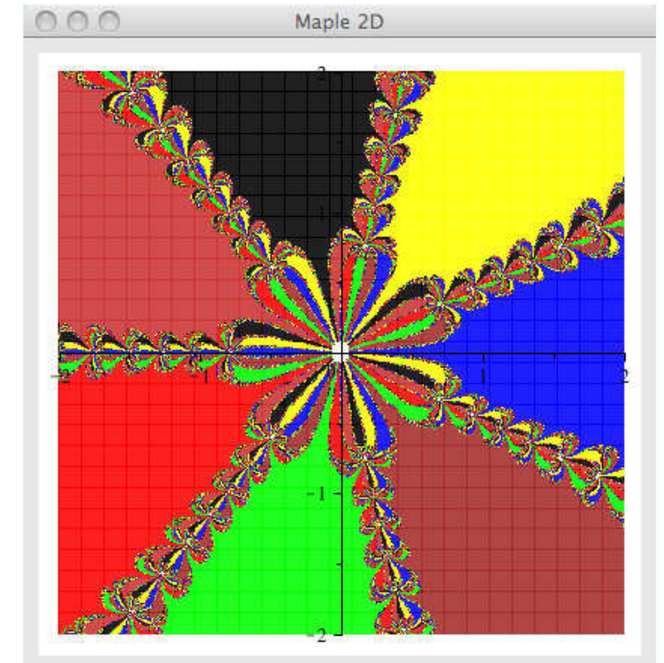
Basins of Attraction

At the correction step, Newton's method can throw the path of course by converging to the wrong root. Unfortunately it is hard to predict if this is going to happen.

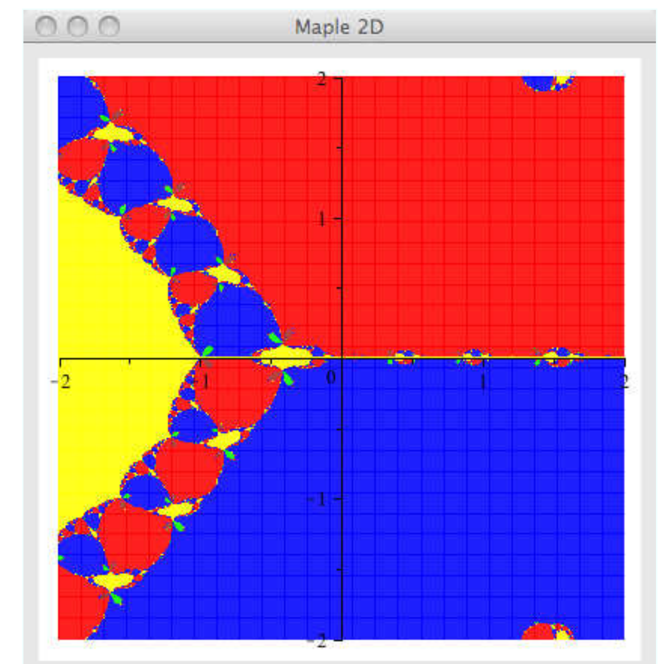
Newton's method will converge to different roots depending on what initial value it is seeded.

Basin of attraction of a root
= { initial points yielding the root }.

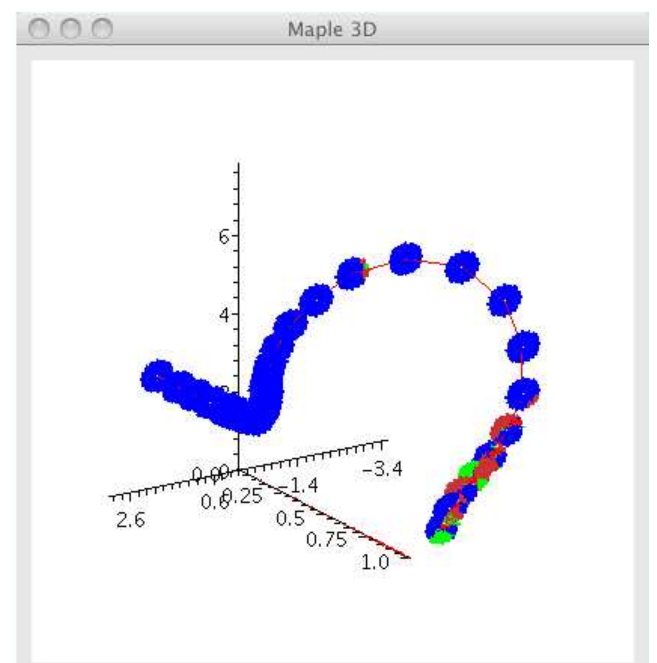
Visualize basins by coloring points in the complex plane by what basin they lie in.



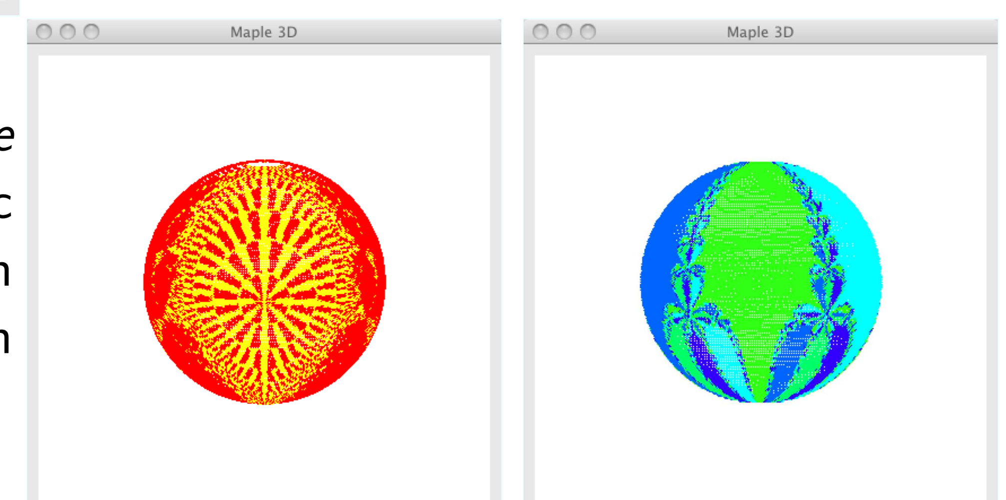
The top right picture is the basin of attraction for $x^7 - 1$ and the bottom right picture is for (1). In both cases we see that the boundaries are fractal in nature, and therefore hard to study.



Below is what we call a **tube of attraction**. For the brown path from the previous section we plot a small disc basin for each time step. Notice how the path stabilized when the basin is dominated by one root.



To visualize the basin for the *entire* complex plane we use a stereographic projection to plot it on the Riemann sphere. From left to right is the basin of attraction for $x^3 - 1$ and $x^5 + 1$.

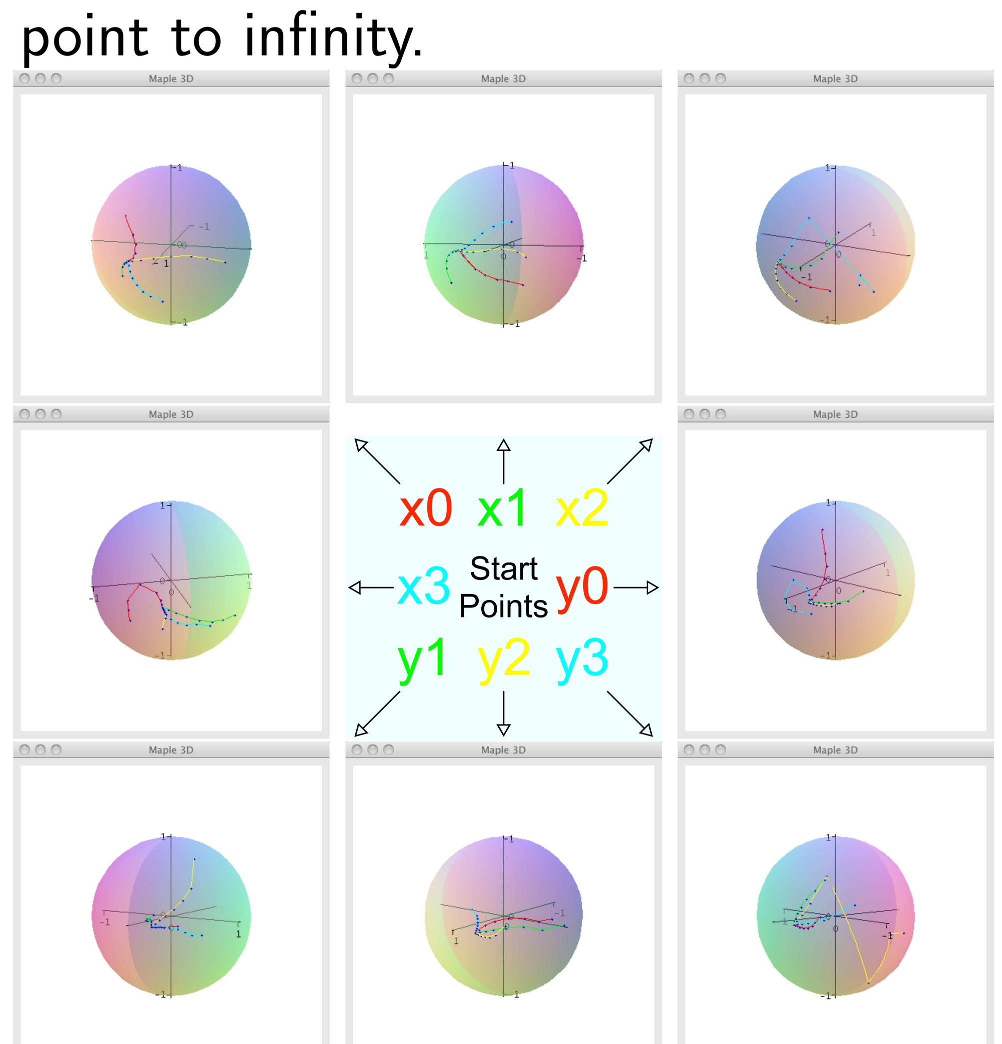


Diverging Paths

A phenomena we do not observe in the univariate case is diverging paths. Consider the target system

$$p(z) = \begin{bmatrix} x^3y + xy^2 + 1 \\ x^4 + xy^2 + 1 \end{bmatrix}$$

Now $\frac{\partial H}{\partial z}$ (the Jacobian) can become singular and our predictor will point to infinity.



To visualize this we again use stereographic projection to plot the paths on a Riemann sphere so we may see paths converge to infinity (the north pole).

The target system $p(z)$ may have up to 16 roots so we must track 16 paths. This is illustrated to the left. All possible pairs (x_i, y_j) with $0 \leq i, j \leq 3$ constitute our 16 start points. Each sphere represents a path that a single component of the solution (x or y) takes. We observe that half the paths diverge.

References

[1] The Numerical Solution of Systems of Polynomials Arising in Engineering and Science, Andrew J. Sommese and Charles W. Wampler II.

[2] Mixed Volume Computation, A Revisit, Tsung-Lin Lee and Tien-Yien Li.