Five Things In Math That I Find Beautiful

Throughout my student career I have been exposed to many (possibly too many) ideas and concepts in mathematics. Although most of the things I have been taught are relevant and useful in their own individual ways, a few things have struck me as sheerly beautiful and amazing. The following is a brief description of zero, Mandelbrot sets, recursion, Hamilton graphs, and the Cauchy Residue Theorem: the five things in math that I find most beautiful.

Zero

The notion of zero is hard to swallow. The typical hang up is the refusal to acknowledge that nothing can reasonably exist. Fortunately, in mathematics we rarely care about the reasonable existence of our theories and ideas.

Zero revolutionized our current number system by offering itself as a place holder. This arguably advanced mathematics by simplifying hand calculations like addition and multiplication, which were a great mess with Roman and Babylonian numerals.

Although we are uncertain about the historical roots of zero many historians trace its origins back to Indian mathematicians in 650AD, who were the first to formally use 0. However, it is clear they borrowed this notion from the Babylonians, which is why historians are hesitant to commit to a certain group.

So even though 0 is a seemingly simple notion in today's standards, without its discovery/invention mathematics would certainly not be the same.

Recursion

"This sentence is false," is quite the logical conundrum. If this sentence was true it would be false, and if it was false it would be true. This whole notion of self reference is really the foundation of mathematical recursion. My interest in this area stems from the rudimentary definitions of many mathematical operations that we take for granted. For instance addition is something that we have always just assumed to be well defined. We teach young children $7+4=11$ and that $1+3=4$, and they believe us, but this is really only a superficial understanding of what addition is. We really have just memorized algorithms knowing that the numbers 0 through 9 pair off to become another number between 0 through 18.

A deeper understanding of the natural number system permits us to define addition primitively. It is first necessary to understand that the natural numbers $\mathbb{N} = \{0, 1, 2, 3...\}$ can be realized to be $\mathbb{N} = \{0, succ(0), succ(succ(0)), ...\}$ where $succ(0)$ denotes the number that comes after 0, after all 7 is just the number that comes after 6 and so on. We then have a definition of addition that looks like the following:

```
add x Zero      = x
add x (Suc y)   = Suc (add x y)
```

Where the second line may be interpreted as `x+(y+1)=(x+y)+1`.

The beautiful thing now is that we can use addition to define a whole whack of mathematical operations. Here are a few basic ones (we will use + instead of the function `add(x,y)` for clarity sake):

```
multiplication
mult x 0    =0
mult x y    =x+mult(x,y-1)

power
pow x 0     =1
pow x y     =mult(x,pow(x,y-1))

factorial
fact 0      =1
fact x      =mult(x,fact(x-1))
```

Using recursion we can take seemingly convoluted definitions and boil them down to a simple base case a recursive step, which I find remarkable.

Mandelbrot Sets

My love for Madelbrot Sets is limited to their sheer aesthetic beauty and my fascination with fractals. A Mandelbrot set marks the set of points in the complex plane such that the corresponding Julia Set, or the set of points z, which do no approach infinity after $R(z) = \frac{P(z)}{Q(z)}$ where $z \in \mathbb{C}^*$ and P and Q are polynomial is repeatedly applied, is connected and not computable by some Turing Machine.

J. Hubbard and A. Douady proved that the Mandelbrot set is connected. Shishukura (1994) proved that the boundary of the Madelbrot set is a fractal with Hausdorff dimension 2. However, it is not yet known if the Madelbrot set is pathwise-connected. To visualize the Mandelbrot set, the limit at which points are assumed to have escaped can be approximated by r_{max} instead of infinity. We can then make a computer generated plot by coloring non-member points depending on how quickly they diverge to r_{max}.

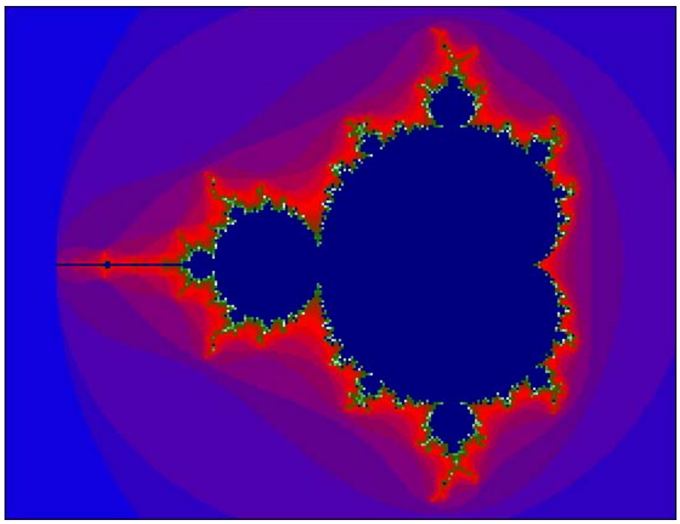

Figure 1: A plot of the Mandelbrot set in which values of $\mathbb{C}$ are colored according to the number of steps required to reach $r_{max} = 2$.

It is easy to see why this plot is considered beautiful by many people.

Hamilton Graphs

I was assigned to work with Prof. Tom Jenkyns at Brock University during my grade 12 year in high school. He introduced me to the notion of graph theory and made it my task to build computer models for arbitrary graphs to which I was instructed to look for paths and circuits in. A path is a traversal along the edges of the graph which visits each node once and a circuit is a path that ends at a node which is connected to the node one left from. To my despair, my programs failed to run on any graph beyond 20 nodes, as I was unaware that this problem is considered NP-complete.

I constructed these graphs by generating all subsets given by selecting k and $k+1$ objects out of $2k+1$ possible objects, these objects became the nodes of the graph. I then connected two nodes, say A and B, with an edge if and only if $A \subset B$ or $B \subset A$. These nodes and edges constituted a Hamilton Graph, which I represented as adjacency matrices.

As I was a naive high school student, I overestimated the power of my computer. I employed a greedy algorithm which simply exhausted all possible routes one could take in a given Hamilton graph, which obviously failed miserably when dealing with graphs of 70 nodes which would have 280 edges and $4^{70} = 1.4 \times 10^{42}$ possible paths.

The problem of proving or disproving that a Hamilton Graph of arbitrary k-value has a Hamiltonian circuit remains an open conjecture. In fact, this problem has so many practical uses that there is a one million dollar award

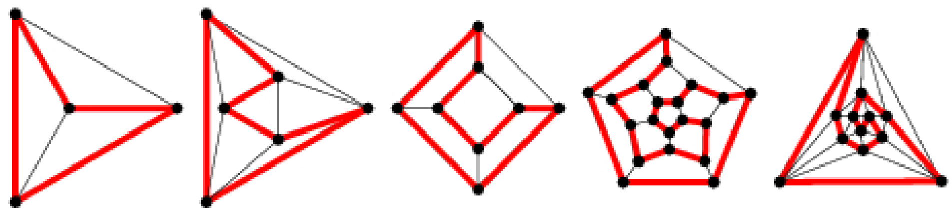

Figure 2: Exhibits Hamilton paths through Hamilton graphs, the graph second from the right has k value 3

for its solution.

Cauchy Residue Theorem

Cauchy Residue theorem is one of the most striking results of complex analysis. The results of this theory can be used as a powerful tool for integrating holomorphic functions round a contour. The statement of the theorem is as follows

Cauchy's residue theorem Let f be a holomorphic inside and on a positively oriented contour except for a finite number of poles, a_1,...,a_N inside γ. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} res\{f(z); a_k\}.$$

To prove this we will need the following definition and lemma:

Lemma (Integration round a pole). Let f be holomorphic inside and on a positively oriented contour γ except at the point a inside γ, where it has a pole of order m. Let

$$f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n$$

be the unique Laurent expansion of f about a. Then

$$\int_\gamma f(z)dz = 2\pi i c_{-1}$$

Residue Suppose that $f \in H(D'(a;r))$ and that f has a pole at a. The residue of f at a is the (unique) coefficient c_{-1} of $(z-a)^{-1}$ in the Laurent expansion of f about a, and is denoted $res\{f(z); a\}$.

Proof of Cauchy's residue theorem Let $f_k(z)$ be the principle part of the Lauren expansion about a_k. Then

$$g := f - \sum_{k=1}^{N} f_k$$

has only removable singularities at $a_1, ..., a_N$; remove them by Cauchy's theorem, $\int_\gamma g(z)dz = 0$. Hence

$$\int_\gamma g(z)dz = \int_\gamma f(z)dz - \sum_{k=1}^{N} f_k(z)dz = 2\pi i \sum_{k=1}^{N} res\{f(z); a_k\},$$

where by the previous lemma, applied to each f_k.