Computer Science 1MD3

Lab 5 – Recurrence Relations

In lab 2 we discussed how functions could be defined recursively. Recall that a recursive definition specifies one or more base cases and a recursive step. The rule for finding the terms that are generated from a functions is called a recurrence relation.

DEFINITION OF RECURRENCE RELATION

A *sequence* is a non-empty set of ordered homogenous elements where a_n refers to the n^{th} term of a sequence. A *recurrence relation* for the sequence $\{a_1 \ldots a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms (i.e. $a_n = a_{n-1} + a_{n-2}$). A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

WORKING WITH RECURENCE RELATIONS

Let $a_n = a_{n-1} - a_{n-2}$ define a recurrence relation, where $a_0 = 3$ and $a_1 = 5$. And suppose that we would like to find the values for a_2 and a_3.

Solution:

$a_2 = a_{2-1} - a_{2-2}$	$a_3 = a_{3-1} - a_{3-2}$
$a_2 = a_1 - a_0$	$a_3 = a_2 - a_1$
$a_2 = 5 - 3$	$a_3 = 2 - 5$
$a_2 = 2$	$a_3 = -3$

Observe how when we substitute a value for n in the recurrence relation the relation become quite clear. The set of all elements generated in this fashion would constitute a sequence that is a solution to this recurrence relation.

VERIFYING SOLUTIONS

Define the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ and set $A = \{a_n \mid a_n = 3n\} = \{0, 1, 6, 9 \ldots 3n\}$. Is set A a solution to the recurrence relation?

Solution:

$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} \\ &= 2\left(3(n-1)\right) - 3\left(n-2\right) \\ &= 6n - 6 - 3n + 6 \\ &= 3n \\ &= a_n \end{aligned}$$

So set A is a solution since it fulfills the criteria of the recurrence relation, however, this is not the only relation. Consider the set $B = \{5, 5, 5, 5, 5, 5, ..., 5\} = \{a_n \mid a_n = 5\}$ does this constitute a solution?

Solution:

$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} \\ &= 2(5) - 3(5) \\ &= 10 - 5 \\ &= 5 \\ &= a_n \end{aligned}$$

As you see set B is also a solution.

SETTING UP RECURRENCE RELATIONS
and why everyone always talks about Fibonacci and his rabbits

Leonardo di Pisa, also known as Fibonacci, was a thirteenth century mathematician who produced an interesting solution to a simple problem. Consider a young pair of rabbits (one of each sex) placed on an island. Rabbits can breed once they are two months old at which time they pair off to give birth to two other rabbits each month. How many rabbits will we have after n months, assuming no rabbits ever die.

Solution:
Let f_n be the number of pairs of rabbits after n months. Since the rabbits do not breed for the first two months we have $f_1 = 1$ and $f_2 = 1$. To find the number of pairs after n months, add the number of rabbits on the island the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2} m since each newborn pair comes from a pair at least two months old. We then are left with the relation:

Fibonacci Sequence

$$f_{n+2} = f_{n-1} + f_{n-2}$$

Which has some very interesting properties such as Φ or the Golan ratio, which is equal to the absolute difference of the two consecutive terms in this sequence as $n \to \infty$.

BACTERIA

Suppose that a population of bacteria, x, triples every hour. How much bacteria do we have after n hours?

Solution:

$$a_1 = x \qquad a_n = 3a_{n-1}$$

Which is intuitively obvious since we will have three times more bacteria then the hour before.

SUMMATION

Consider $\sum_{k=1}^{n} (2k - 1)$, where we would like to instead have a recursive definition. As we know we could re-write this as $2(n) - 1 + \sum_{k=1}^{n-1} (2k - 1)$ so the recursive definition would follow from this.

$$\sum_{k=1}^{1} (2k - 1) = 2(1) - 1 = 1 \rightarrow a_1 = 1$$

$$\sum_{k=1}^{n} (2k - 1) = a_n$$

$$2(n) - 1 + \sum_{k=1}^{n-1} (2k - 1) = 2(n) - 1 + a_{an-1}$$

$$a_n = 2(n) - 1 + a_{an-1}$$