§ ALTERNATING SERIES TEST

The series
$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$$
is called the alternating harmonic series.

Recall: $\sum |a_n| = \sum \frac{1}{n}$ is the (divergent) harmonic series.

Prop$^{\underline{n}}$ The alternating harmonic series converges.

Proof:
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$
$$= 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2K} - \frac{1}{2K+1}\right) - \cdots$$
$$= 1 - \frac{1}{2\cdot 3} - \frac{1}{4\cdot 5} - \cdots - \frac{1}{(2K)(2K+1)}$$
$$= 1 - \sum_{K=1}^{\infty} \frac{1}{(2K)(2K+1)} \quad (*)$$

But $\sum_{K=1}^{\infty} \frac{1}{(2K)(2K+1)} < \sum_{K=1}^{\infty} \frac{1}{K^2}$ (the convergent p-series.) Thus $(*)$ also converges $\square$

Thm Alternating Series Test (AST)

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges when the following are satisfied.

1. $u_n \geq 0$ 2. $u_{n+1} \leq u_n$ "eventually"
3. $\lim_{n\to\infty} u_n = 0$.

EXAMPLE $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \Rightarrow u_n = \frac{1}{n}$

1. $\frac{1}{n} \geq 0 \Rightarrow u_n \geq 0$

2. $\frac{1}{n+1} \leq \frac{1}{n} \Rightarrow u_{n+1} \leq u_n$

3. $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

By AST $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent.

§ Conditional Convergence

Def$\underline{^n}$ Condition Convergence

A convergent series that does not absolutely converge is conditionally convergent.

EXAMPLE The alternating harmonic series is conditionally convergent.

EXAMPLE $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} = (*)$ abs con? cond con?

Recall: $\sum \frac{1}{n \ln n}$ diverges by integral test.

(i.e. $\int_2^{\infty} \frac{1}{x \ln x} dx$ is divergent) $\Rightarrow$ not abs con.

Perform AST: (1) $n \geq 2 \Rightarrow n \cdot \ln n \geq 0$

$\hookrightarrow u_n = \frac{1}{n \ln n}$ $\Rightarrow \frac{1}{n \ln n} \geq 0 \Rightarrow u_n \geq 0$

(2) $\frac{1}{(n+1)\ln(n+1)} \leq \frac{1}{n \ln n} \Rightarrow u_{n+1} \leq u_n$

(3) $\lim_{n \to \infty} \frac{1}{n \ln n} = 0$ So $(*)$ is conditionally convergent.

EXAMPLE $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = (*)$ con/div ?
abs/cond ?

$$\sum_{n=0}^{\infty} \left| (-1)^n \frac{1}{2^n} \right| = \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2.$$

thus $(*)$ converges absolutely

EXAMPLE $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ con/div? abs/cond?

- $(-1)^{n+1} \frac{n^2}{n^3+1} \xrightarrow{n \to \infty} 0.$ $\Rightarrow$ Convergence is possible

- ABSOLUTE CONVERGENCE?

(Do LCT w/ $\sum \frac{1}{n}$.) Notice $\lim_{n \to \infty} \left| \frac{n^2}{n^3+1} \Big/ \frac{1}{n} \right|$

$= \lim_{n \to \infty} \left| \frac{n^3}{n^3+1} \right| = 1.$ By LCT $\sum \frac{1}{n}$ & $\sum \frac{n^3}{n^3+1}$ both diverge.

- CONDITIONAL CONVERGENCE?

$u_n = n^2 / n^3+1$ ① $n \geq 1 \Rightarrow u_n \geq 1.$

② Show $u_{n+1} \leq u_n$. Notice: $f(x) = x^2 / x^3+1$ has $f(n) = u_n$. $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$ and

$f'(x) < 0$ when $x(2-x^3) < 0 \Rightarrow 2 - x^3 < 0$
$\Rightarrow x^3 > 2 \Rightarrow x = \sqrt[3]{2}.$

Thus f is decreasing for $x > \sqrt[3]{2}$

$\Rightarrow \quad f(2) \geq f(3) \geq f(4) \geq f(5) \cdots$

$\Rightarrow \quad u_2 \geq u_3 \geq u_4 \geq u_5$

$\Rightarrow \quad u_{n+1} \leq u_n$

$\Rightarrow$ P2.

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0 \Rightarrow \boxed{\text{P3.}}$$

By AST $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2}{n^3+1}$ converges.

EXERCISE ⑦ $= 4\sum_{K=0}^{\infty} \frac{(-1)^K}{K+7} = \frac{4}{7} - \frac{4}{8} + \frac{4}{9} - \frac{4}{10} + \frac{4}{11} - \cdots$

No abs conv. Compare w/ $\sum \frac{1}{K}$ using LCT.

But: $u_K = \frac{1}{K+7} \Rightarrow u_K \geq 0$ ①

$u_{K+1} \leq u_K \Leftrightarrow \frac{1}{K+8} \leq \frac{1}{K+7} \Leftrightarrow K+7 \leq K+8$

$\Leftrightarrow K \leq K+1 \Leftrightarrow$ TRUE. ②

$\lim_{K\to\infty} \frac{1}{K+7} = 0.$ ③

Convergent by AST

Not abs conv.

⑦ is conditionally convergent

<u>EXERCISE</u> $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{10^n}$

$(-1)^n \frac{n}{10^n} \to 0$ So conv is <u>possible</u>

Note $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10^n}$ is <u>conv</u> because

$$\lim_{n\to\infty} \left(\frac{n}{10^n}\right)^{\frac{1}{n}} = \frac{1}{10} < 1.$$

and Root-Test.

Thus $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{10^n}$ is convergent.

EXERCISE

- $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{n}{5}\right)^n$

- $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

- $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$