## § RECAP

- $\sum_{n=0}^{\infty} a_n$ can converge ($<\infty$) or diverge ($\geq \infty$)

- $\sum_{n=0}^{\infty} \frac{1}{n}$ (harmonic series) diverges

- $\sum_{n=0}^{\infty} \left(\frac{1}{n}\right)^p$ ($p$-series) converges for $p > 1$.

- $\sum_{n=0}^{\infty} r^n$ (geometric series) converges for $|r| < 1$

and $= \frac{1}{1-r}$

## Integral Test

$$\int_a^\infty f(x)\,dx \quad \& \quad \sum_{n=a}^{\infty} f(n)$$

diverge and converge together.

## Limit Comparison Test

$0 \leq d_n \leq a_n \leq c_n$

- $\sum c_n$ con $\Rightarrow$ $\sum a_n$ con
- $\sum d_n$ div $\Rightarrow$ $\sum a_n$ div

# $\int$ ABSOLUTE CONVERGENCE

Suppose $\{a_K\}$ contains negative terms like in the series:

$$\sum a_K = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$$

Notice: $\sum |a_K| = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2$ is convergent.

**Def^n Absolute Convergence**

$\sum a_n$ "converges absolutely" when

$\sum |a_n|$ converges.

**Thm** $\sum |a_K|$ converges $\Rightarrow \sum a_K$ converges

"Proof" $-\sum |a_K| \leq \sum a_K \leq \sum |a_K|$

**EXERCISE:** Note $\sum |a_n|$ div $\not\Rightarrow \sum a_n$ div.

Find $a_n$ so that $\sum |a_n|$ div and $\sum a_n$ con.

EXAMPLE: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ is a convergent p-series.

EXAMPLE: $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \cdots$

converges because

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{|n^2|} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$ is a convergent p-series

## Ratio Test

For $\sum a_n$ we investigate $\frac{a_{n+1}}{a_n}$. to determine convergence. For instance

$\sum br^n$ ~~$\sum br^n$~~ has $\frac{a_{n+1}}{a_n} = \frac{br^{n+1}}{br^n} = r$

and converges when $r<1$.

We can generalize this result.

**Thm** Ratio Test

Let $\sum a_n$ be a series and suppose that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$$

(a) $\boxed{\rho<1} \Rightarrow \sum a_n$ converges

(b) $\boxed{\rho>1} \Rightarrow \sum a_n$ diverges

(c) $\boxed{\rho=1} \Rightarrow$ INCONCLUSIVE

* Recall: $x \to y$ when $\lim_{n\to\infty} x = y$. *



"Proof" Suppose $\left|\frac{a_{n+1}}{a_n}\right| \to \rho < 1$

let $r \in [\rho, 1] \Rightarrow r - \rho > 0 \Rightarrow \exists \varepsilon : r - \rho = \varepsilon$

For $n \geq N$ (i.e. eventually)

$$\left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon = r$$

Thus...

$$|a_{N+1}| < r|a_N|$$

$$|a_{N+2}| < r|a_{N+1}| < r \cdot r|a_N|$$

$$|a_{N+3}| < r|a_{N+2}| < r \cdot r \cdot r|a_N|$$

$$\vdots$$

$$|a_{N+m}| < r|a_{N+m-1}| < r^m|a_N|$$

Adding all rows:

$$\sum_{m=N}^{\infty} |a_m| = \sum_{m=0}^{\infty} |a_{N+m}| < |a_N| \sum_{m=0}^{\infty} r^m$$

Thus $\sum |a_n|$ converges when $|r| < 1$.

*note* $xy! = x \cdot y \cdot (y-1) \cdot (y-2) \cdot \cdots 1$

$(xy)! = xy \cdot (xy-1) \cdot (xy-2) \cdot \cdots 1$

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**EXAMPLE** $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} \Rightarrow a_n = \frac{2n!}{n!n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2(n+1)!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{2n!} = \frac{2(n+1)n!}{(n+1)n!(n+1)n!} \cdot \frac{n!n!}{2n!}$$

$= \frac{(n+1)}{(n+1)(n+1)} = \frac{1}{n+1}$ and so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$< 1 \Rightarrow \sum a_n$ is convergent

**EXAMPLE** (Inconclusive) $\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow a_n = \frac{1}{n} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} \to 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow a_n = \frac{1}{n^2} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n^2)}{(n+1)^2} \to 1$

But $\sum \frac{1}{n^2}$ converges & $\sum \frac{1}{n}$ diverges!

## § Root Test

**thm** Root Test

Let $\sum a_n$ be a series and suppose that

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho.$$

(a) $\boxed{\rho<1} \Rightarrow \sum |a_n|$ is convergent

(b) $\boxed{\rho>1 \text{ or } \rho=\infty} \rightarrow \sum a_n$ is divergent

(c) $\boxed{\rho=1}$ INCONCLUSIVE

**EXAMPLE** $a_n = \begin{cases} n/2^n & n \text{ odd} \\ 1/2^n & n \text{ even.} \end{cases}$ Does $\sum a_n$ converge?

Notice: $\frac{1}{2^n} \leq a_n \leq \frac{n}{2^n}$

$\rightarrow \left(\frac{1}{2^n}\right)^{\frac{1}{n}} \leq a_n^{\frac{1}{n}} \leq \left(\frac{n}{2^n}\right)^{\frac{1}{n}}$

sandwich lemma $\Rightarrow$ $\frac{1}{2} \leq \lim_{n\to\infty} a_n^{\frac{1}{n}} \leq \frac{1}{2} \cdot \lim_{n\to\infty} n^{\frac{1}{n}}$

Recall $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1.$

Thus $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2} < 1 \Rightarrow \sum |a_n|$ conv. by root-test

$\Rightarrow \sum a_n$ converges

<u>EXERCISE</u>: $\displaystyle\sum_{n=1}^{\infty} \left( \frac{1-\sqrt{3n}}{(2n+1)^{\frac{1}{2}}} \right)$ con/div?

$$\Rightarrow a_n = \left( \frac{1-(3n)^{\frac{1}{2}}}{(2n+1)^{\frac{1}{2}}} \right)^n$$

Root-test: $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \left| \frac{1-(3n)^{\frac{1}{2}}}{(2n+1)^{\frac{1}{2}}} \right|$

$$= \lim_{n\to\infty} \frac{(3n)^{\frac{1}{2}}-1}{(2n+1)^{\frac{1}{2}}} = \lim_{n\to\infty} \frac{(3n)^{\frac{1}{2}}}{(2n)^{\frac{1}{2}}} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{3}{2}} > 1.$$

By root-test $\sum \cancel{|}a_n\cancel{|}$ is divergent.

EXERCISE $\sum_{n=1}^{\infty} \frac{n!}{n^n} \Rightarrow a_n = n!/n^n$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)\, n! \cdot n^n}{(n+1)(n+1)^n n!}$$

$$= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \, \frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = 1/e.$$

Recall $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$

Thus $\sum |a_n|$ is conv. by Ratio Test.

$\Rightarrow \sum |a_n|$ converges.

$(2n)! \neq 2n!$

**EXERCISE** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \Rightarrow a_n = \frac{(2n)!}{(n!)^2} = \frac{(2n)!}{n!\,n!}$

$$\frac{a_{n+1}}{a_n} = \frac{(2(n+1))!}{(n+1)!\,(n+1)!} \cdot \frac{n!\,n!}{(2n)!}$$

$$= \frac{(2n+2)!\,n!\,n!}{(n+1)\,n!\,(n+1)\,n!\,(2n)!} = \frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)(2n)!}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$= \lim_{n\to\infty} \frac{4n^2}{n^2} = 4.$$

$\rightarrow \sum a_n$ diverges by Ratio Test.