§ Infinite Series

A **series** is a sum of numbers:

SERIES $a_0 + a_2 + \cdots + a_K$ w/ $a_i \in \mathbb{R}$

INFINITE SERIES $a_0 + a_1 + \cdots$

PARTIAL SUM $S_K = a_0 + \cdots + a_K = \sum_{i=0}^{K} a_i$

INFINITE SUM $S = a_0 + \cdots + a_K + \cdots = \sum_{i=0}^{\infty} a_i = \sum a_i$

empty limits implies infinite.

We are interested in two things:

1. Convergence/Divergence behaviour
2. The exact/closed form.

WARNING: Take care with indexes. We often (out of neccesity) switch from $K=0$ to $K=1$ as the lower bound.

There no general method for finding the exact sum of arbitrary sums/series.

We are usually satisfied to find an approx or lower/upper bounds

Something like $\sum \frac{1}{2K}$ isnt a closed for because we cannot compute w/ it.

QUESTION Is $\sum_{K=1}^{\infty} \frac{1}{2^K}$ divergent or convergent?

If convergent what is the "closed form" or precise sum.

$$S_1 = \frac{1}{2^1} \qquad = 1 - 1/2$$

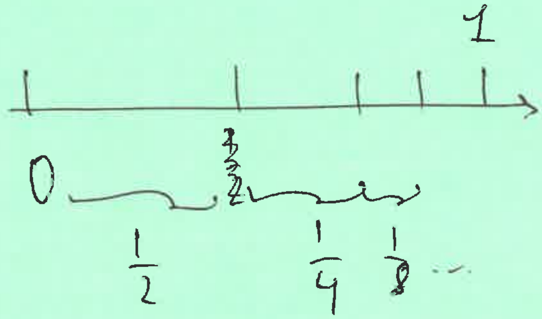
$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \qquad = 1 - \frac{1}{4}$$

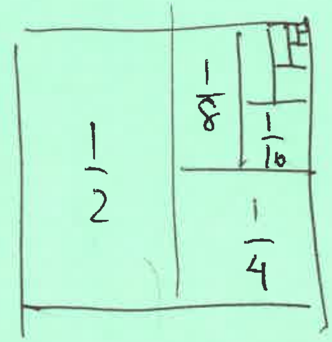
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \qquad = 1 - \frac{1}{8}$$

$$\Rightarrow \underbrace{S_K = 1 - \frac{1}{2^K}}_{\text{(closed form)}} \Rightarrow S = \lim_{K \to \infty} 1 - \frac{1}{2^K} = 1 - 0 = 1.$$

Thus $\sum_{K=1}^{\infty} \frac{1}{2^K}$ converges to 1. (precise sum)

Visually!





Def$^{\underline{n}}$ Geometric Series

$$\sum_{k=0}^{\infty} ar^k = ar^0 + ar^1 + \cdots$$

w/ $a, r \in \mathbb{R}$ and $|r| < 1$.

EXAMPLE $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^k$ has $a = \frac{1}{2}$ and $r = \frac{1}{2}$.

Thm When $|r| < 1$

$$\sum_{n=0}^{\infty} c \cdot r^n = \frac{c}{1-r} < \infty$$

and it <u>diverges</u> otherwise.

Proof: $S_n = a + ar + ar^2 + \cdots + ar^n$

$$\Rightarrow rS_n = \overset{ar}{\cancel{ra}} + ar^2 + ar^3 + \cdots + ar^{n+1}$$

Thereby $S_n - rS_n = a - ar + ar - ar^2 + ar^2 + \cdots - ar^{n+1}$

$$= a - ar^{n+1}$$

Thereby $S_n(1-r) = a(1-r^{n+1})$

$$\Rightarrow \quad S_n = \frac{a(1-r^{n+1})}{(1-r)}$$

and $$\lim_{n\to\infty} S_n = \frac{a(1-0)}{(1-r)} = \frac{a}{1-r}$$

provided $|r|<1$.

<u>EXAMPLE</u> $$\sum_{K=0}^{\infty} \frac{1}{2^K} = \sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^K = \frac{1}{1-\frac{1}{2}} = 2.$$

<u>EXAMPLE</u> $$\sum_{K=1}^{\infty} \left(\frac{1}{3}\right)^{K+1} = \sum_{K=0}^{\infty} \left(\frac{1}{3}\right)^{K+2} = \frac{1}{9}\sum_{K=0}^{\infty} \left(\frac{1}{3}\right)^K$$

$$= \frac{1}{9} \cdot \frac{1}{1-\frac{1}{3}} = \frac{3}{2 \cdot 9} = \frac{1}{6}.$$

<u>EXAMPLE</u> $$\sum_{K=0}^{\infty} \frac{(-1)^n \cdot 5}{4^n} = 5\sum_{K=0}^{\infty} \left(\frac{-1}{4}\right)^n$$

$$= 5 \cdot \frac{1}{1-\left(\frac{-1}{4}\right)} = 4.$$

EXAMPLE What fraction has decimal expansion $5.\overline{23}\,\overline{23}\,\overline{23}$?

$$5.232323\ldots = 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(1000)^3} + \cdots$$

$$= 5 + \sum_{K=1}^{\infty} \frac{23}{100^K} \quad \cancel{5 + \frac{23}{100}\sum_{K=0}^{\infty}\left(\frac{1}{100}\right)^K}$$

$$= 5 + \frac{23}{100}\sum_{K=0}^{\infty}\left(\frac{1}{100}\right)^K = 5 + \frac{23}{100}\cdot\frac{1}{1-\frac{1}{100}}$$

$$= 5 + \frac{23}{99} = \frac{518}{99}.$$

Def$^{\underline{n}}$ Telescoping Series

A telescoping series is one where "most" terms cancel, simplifying the sum.

__EXAMPLE__ $\displaystyle\sum_{K=1}^{\infty} \frac{1}{K(K+1)} = \sum_{K=1}^{\infty} \frac{1}{K} - \frac{1}{K+1}$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \cdots + \left(\cancel{\tfrac{1}{K}} - \frac{1}{K+1}\right)$$

$$= 1 - \frac{1}{K+1} \qquad \text{(a "closed form")}$$

Thereby $\displaystyle\sum_{K=1}^{\infty} \frac{1}{K(K+1)} = \lim_{K\to\infty} 1 - \frac{1}{K+1} = 1.$

§ Divergence Testing

Suppose $S_n = \displaystyle\sum_{K=0}^{n} a_K$ and $\displaystyle\lim_{K\to\infty} a_K = L > 0$.

This means $S = \cdots + L + L + L + \cdots$ which __diverges__.

__Thm__ ($N^{\underline{th}}$ Diverg~~ence Test~~) $\displaystyle\sum_{n=0}^{\infty} a_n$ converges $\Rightarrow$ $\displaystyle\lim_{n\to\infty} a_n = 0$.

Notice the contrapositive is

$$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ diverges}$$

We do not have

$$\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=0}^{\infty} a_n \text{ converges}$$

EXAMPLE $\sum_{n=0}^{\infty} n$ diverges. as $n \to \infty \neq 0$

EXAMPLE Consider $\sum_{K=1}^{\infty} \ln\left(\frac{K+2}{K+1}\right)$. and notice

$\lim_{K\to\infty} \ln\left(\frac{K+2}{K+1}\right) = 0$. what does n^{th} divergence tells us about the sum?

NOTHING! $\sum_{K=1}^{\infty} \ln\left(\frac{K+2}{K+1}\right) = \sum \ln(K+2) - \ln(K+1)$

So

$$S_1 = \ln(3) - \ln(2)$$

$$S_2 = \ln(3) - \ln(2) + \ln(4) - \ln(3) = \ln(4) - \ln(2)$$

$$S_3 = \ln(4) - \ln(2) + \ln(5) - \ln(4) = \ln(5) - \ln(2)$$

$$\vdots$$

$S_K = \ln(K+2) - \ln(2) \Rightarrow S_K \to \infty$ and diverges.

EXERCISE:

- $$\sum_{K=1}^{\infty} \frac{3^{K-1}-1}{6^{K-1}}$$
- $$\sum_{K=0}^{\infty} \frac{4}{2^n}$$

EXERCISE: Find a "closed form" (i.e. an equation for)

- $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1} 2^{n-1} + \cdots$
- $\frac{5}{1 \cdot 2} + \frac{5}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

EXERCISE: which series converge/diverge

- $$\sum_{K=0}^{\infty} \frac{2^K + 3^K}{4^K}$$
- $$\sum_{K=0}^{\infty} \frac{3^K + 4^K}{3^K + 4^K}$$
- $$\sum_{K=0}^{\infty} \ln\left(\frac{K}{2K+1}\right)$$
- $$\sum_{K=0}^{\infty} \frac{e^{K\pi}}{\pi^{Ke}}$$