§ SEQUENCES

EXAMPLE: 1, 2, 4, 8, 16, 32, ...

Is an infinite sequence of powers of two.

By designating the n^{th} term of this sequence by a_n so that

$$a_n = 2^n$$

we can say the sequence is denoted $\{a_n\}$

Defn Sequence

$\{a_n\}$ denotes an infinite sequence w/ first index $n = 0$.

EXAMPLE: $b_1 = \frac{1}{1}$, $b_2 = \frac{1}{2}$, $b_3 = \frac{1}{4}$, $b_4 = \frac{1}{8}$, ...

Then $b_n = \frac{1}{2^n}$ for $n = 1, 2, 3, \ldots$ and the sequence is denoted $\{b_n\}$.

The limit of a sequence at infinity is the "value" (if it exists) of "b_∞" denoted:

$$\lim_{n \to \infty} b_n = L \quad \text{OR} \quad b_n \to L$$

In this case $\lim_{n \to \infty} \frac{1}{2^n} = 0$ and so the sequence is said to be convergent.

<u>QUESTION</u> Let $d_n = \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n+1)}$

What is $\lim\limits_{n\to\infty} d_n$?

$$d_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$$

$$d_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{4}{6} = \frac{2}{3}$$

$$d_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{9}{12} = \frac{1}{4}$$

<u>NOTICE</u>: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

and that

$$d_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 \underbrace{}_{+0} \underbrace{}_{+0} \underbrace{}_{+0} \underbrace{}_{+0} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}.$$

and thus $\lim\limits_{n\to\infty} d_n = 1 - 0 = 1.$

<u>Thm</u> Let
- $\{a_n\}$ and $\{b_n\}$ be sequences of $\mathbb{R}$.
- $A, B \in \mathbb{R}$
- $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$ then:

1. $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$ (and sub.)
2. $\lim\limits_{n\to\infty} K \cdot a_n = K \cdot \lim\limits_{n\to\infty} a_n = K \cdot A$ for $K \in \mathbb{R}$
3. $\lim\limits_{n\to\infty} a_n \cdot b_n = A \cdot B$ (and division)

EXAMPLE What $\{a_n\}$ gives...

- $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \cdots, \quad (-1)^{n+1}\frac{1}{n^2} = \{a_n\}$

- $\frac{5}{1}, \frac{8}{2}, \frac{11}{6}, \frac{14}{24}, \frac{17}{120}, \cdots, \quad \frac{2+3n}{n!} : n \geq 1.$

- $\frac{1}{9}, \frac{2}{12}, \frac{2^2}{15}, \frac{2^3}{18}, \frac{2^4}{21}, \cdots, \quad \frac{2^n}{3\cdot(3+n)} : n \geq 0$

Thm: Sandwich Thm for Sequences

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be real sequences.

If there is $N \in \mathbb{N}$ such that $\forall n \geq N$

$$a_n \leq b_n \leq c_n$$

then

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \implies \lim_{n\to\infty} b_n = L.$

QUESTION $\lim_{n\to\infty} \frac{\sin n}{n}$

Notice $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ so that

$$\lim_{n\to\infty} \frac{-1}{n} = 0 \leq \lim_{n\to\infty} \frac{\sin n}{n} \leq 0 = \lim_{n\to\infty} \frac{1}{n} = 0$$

(~~[illegible]~~ (we can apply limits across inequalities.)

Thm Let $\{a_n\}$ be a real sequence, then if

- $a_n \to L$
- f is a function continuous at L.
- $f(a_n)$ is defined for all a_n.

i.e. There is a continuous function, f, passing through each $\{(1,a_1),(2,a_2),\ldots\}$

then $f(a_n) \longrightarrow f(L)$

OR EQUIV

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

EXAMPLE: Show that $\left(\frac{n+1}{n}\right)^{\frac{1}{2}} \to 1$

i.e. $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{\frac{1}{2}} = 1$. Let $a_n = \frac{n+1}{n} \Rightarrow \lim_{n\to\infty} a_n = 1$.

Notice $f(x) = \sqrt{x}$ satisfies the last Thm.

Thus $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(1) = 1$.

$\Rightarrow \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{\frac{1}{2}} = 1$.

In order to use L'Hopitals rule we have:

<u>Thm</u> Let $f(x)$ be a real function and $\{a_n\}$ a real sequence. Provided $\exists N \in \mathbb{N}$ such that

- $\forall x \geq N$; $f(x)$ is defined.
- $a_n = f(n)$ for $n \geq N$.

i.e. f passes through $(1, a_1), (2, a_2), (3, a_3), \ldots$

then $\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$.

(Basically, if we can find a function where "eventually" $f(n) = a_n$ for $n \in \mathbb{N}$.

EXAMPLE

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \quad \text{by last thm.}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0.$$

§ USEFUL IDENTITIES "$A \to B$" means $\lim_{n\to\infty} A = B$.

- $\frac{\ln n}{n} \to 0$
- $\sqrt[n]{n} \to 1$
- $x^{\frac{1}{n}} \to 1$ for $x > 0$
- $x^n \to 0$ for $|x| < 1$
- $\left(1 + \frac{x}{n}\right)^n \to e^x$
- $\frac{x^n}{n!} \to 0$

QUESTION $C_n = \dfrac{2e^n + e^{-n}}{e^n + 3e^{-n}}$

What is $\lim\limits_{n\to\infty} C_n = ?$

~~By Thm~~ $\lim\limits_{n\to\infty} C_n = \lim\limits_{x\to\infty} f(x)$ for $f(x) = \dfrac{2e^x + \frac{1}{e^x}}{e^x + \frac{3}{e^x}}$

and thus $\lim\limits_{n\to\infty} C_n = \lim\limits_{x\to\infty} \dfrac{\frac{1}{e^x}}{\frac{1}{e^x}} \dfrac{2e^x + \frac{1}{e^x}}{e^x + \frac{3}{e^x}}$

$$= \lim_{x\to\infty} \frac{2 + \frac{1}{e^{2x}}}{1 + \frac{3}{e^{2x}}} = 2.$$

by Thm

§ Bounded and Monotonic

Def$^{\underline{n}}$ $\{a_n\}$ is "bounded from above"

when $\exists M$, called the upper bound, such that

$a_0 \leq M, \quad a_1 \leq M, \quad a_2 \leq M, \ldots$

the smallest upper bound is called the least upper bound.

Def$^{\underline{n}}$ $\{a_n\}$ is "bounded from below"

when $\exists M$, called the lower bound, such that

$a_0 \geq M, \quad a_1 \geq M, \quad a_2 \geq M, \ldots$

the largest upper bound is called the greatest upper bound.

Def$^{\underline{n}}$ $\{a_n\}$ is "bounded"

When $\{a_n\}$ is "bounded" from above and below and is "unbounded" otherwise.

EXAMPLE • $a_n = n$ is not bounded above but is bounded below by 0. for $n \in \mathbb{N}$.

• $a_n = \frac{n}{n+1}$ is bounded above by 1 and below by $\frac{1}{2}$.

Def$\underline{^n}$ $\{a_n\}$ is non-decreasing when

$a_n \leq a_{n+1}$ i.e. not strictly increasing

Def$\underline{^n}$ $\{a_n\}$ is non-increasing when

$a_n \geq a_{n+1}$ i.e. not strictly decreasing

Def$\underline{^n}$ $\{a_n\}$ is monotonic if it is either nondecreasing ~~&~~ or nonincreasing.

EXAMPLE
- $1, 2, 3, \dots, n, \dots$ is nondecreasing
- $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is non-decreasing
- $3, 3, 3, \dots$ both non-dec & non-inc.
- $1, -1, 1, -1, \dots$ not monotonic.

Thm Monotonic Sequence Thm

If $\{a_n\}$ is both bounded & monotonic then it converges.

EXAMPLE Does $\frac{(-1)^{n+1}}{n}$ converge?

Yes. Is it monotonic? No.

EX Find ~~the~~ $a_1, a_2, a_3, a_4 \dots$

1. $a_n = \frac{1-n}{n^2}$ 2. $a_n = \frac{2^n - 1}{2^n}$

EX Guess the pattern. Write a_n.

1. $1, -4, 9, -16, 25, \dots$
2. $0, 1, 1, 2, 2, 3, 3, 4, 4, \dots$
3. $-\frac{3}{2}, -\frac{1}{6}, \frac{1}{12}, \frac{3}{20}, \frac{5}{30}, \dots$

EX Find the limit.

1. $a_n = \frac{2n+1}{1-3\sqrt{n}}$ 2. $a_n = \frac{n+3}{n^2+5n+6}$

3. $a_n = \left(1 - \frac{1}{n^2}\right)^n$ 4. $a_n = \sqrt[n]{4^n \cdot n}$

5. $a_n = \sqrt{\frac{2n}{n+1}}$ 6. $a_n = \frac{n!}{2^n \cdot 3^n}$