§ Partial Fraction Expansion

Motivation

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

where $\int \frac{x+5}{x^2+x-2}\,dx$ is seemingly harder than

$\int \frac{2}{x-1} - \frac{1}{x+2}\,dx$ where we generate logarithms

Fundamental Thm Algebra

Every real-valued polynomial can be factored into a product of linear and irreducible quadratics.

Note: Irreducible quadratics are those w negative discriminant.

EXAMPLE $X^2+X-2 = (X+2)(X-1)$ Reducible

X^2+1 Irreducible ... over $\mathbb{R}$

EXAMPLE $2x^3+3x^2-2x = x(2x^2+3x-2)$

$= x(2x-1)(x+2)$

is the "factorization" into irreducibles

Def^n^ Proper

A "proper polynomial fraction" is one of the form

$$\frac{a_n x^n + \cdots + a_1 x + a_0}{a_m x^m + \cdots + a_1 x + a_0}$$

where $m > n$.

We can always use long division to obtain a proper polynomial fraction from an improper one.

In the complex numbers $\mathbb{C}$ where $i^2=-1$

$$X^2+1 = (x+i)(x-i) \;!$$

EXAMPLE: $\int \frac{x^3+x}{x-1}\,dx = \int (x^2+x+2) + \frac{2}{x-1}\,dx = ⊛$

$$
\begin{array}{r}
(x^2 + x + 2)\ R\ 2 \\
x-1\ \overline{)\ x^3 + 0x^2 + x + 0} \\
\underline{x^3 - x^2 \quad\quad\quad} \\
x^2 + x \quad \\
\underline{x^2 - x \quad} \\
2x \quad \\
\underline{2x - 2} \\
2
\end{array}
$$

$⊛ = x^3/3 + x^2/2 + 2x + 2\ln|x-1| + C$

§ Distinct Linear Factors

"Roots"

Form: $Q(x) = (a_0x+b_0)(a_1x+b_1)\cdots(a_kx+b_k)$

and no roots / factors repeat.

Then for $\deg R(x) < \deg Q(x)$

$$\frac{R(x)}{Q(x)} = \frac{A_0}{a_0x+b_0} + \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_k}{a_kx+b_k}$$

w/ $A_0, \ldots, A_k \in \mathbb{R}$.

EXAMPLE: $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = (*)$

$$2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$$

$$(*) \quad \frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$\Rightarrow \quad A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$
$$= x^2 + 2x - 1$$

$$\Rightarrow \begin{cases} 2A + B + 2C = 1 \\ 3A + 2B - C = 2 \\ -2A = -1 \end{cases} \Rightarrow A = \frac{1}{2}$$

$$\Rightarrow \begin{cases} B + 2C = 0 \\ 2B - C = \frac{1}{2} \end{cases} \Rightarrow 5B = 1 \Rightarrow B = 1/5$$
$$\Rightarrow C = -1/10.$$

$$\text{So } (*) = \int \frac{1/2}{x} + \frac{1/5}{2x-1} + \frac{-1/10}{x+2}\, dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

EXAMPLE: $\displaystyle\int \frac{1}{x^2-a^2}\,dx = *$

$$\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$A(x+a) + B(x-a) = 1$$

$$\Rightarrow \begin{cases} A+B=0 \\ aA - aB = 1 \end{cases} \Rightarrow aA - a(-A) = 1$$

$$\Rightarrow 2aA = 1 \Rightarrow A = \frac{1}{2a} \Rightarrow B = \frac{-1}{2a}$$

$$* = \int \frac{1/2a}{x-a} + \frac{-1/2a}{x+a}\,dx$$

$$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

§Repeated Linear Factors

FORM:

$$\frac{R(x)}{(a_0x+b_0)^l} = \frac{A_1}{a_0x+b_0} + \frac{A_2}{(a_0x+b_0)^2} + \cdots + \frac{A_l}{(a_0x+b_0)^l}$$

EXAMPLE:

$$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

EXAMPLE: $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = (*)$

Integrand is improper.–

$$\begin{array}{r|l}
 & (x+1)\ R\ \ 4x \\
x^3-x^2-x+1 & x^4+0x^3-2x^2+4x+1 \\
 & x^4-x^3-x^2+x \\
\hline
 & x^3-x^2+3x+1 \\
 & x^3-x^2-x+1 \\
\hline
 & 4x
\end{array}$$

$$(*) = \int (x+1) + \frac{4x}{x^3-x^2-x+1} dx$$

partial fractions...

$$x^3-x^2-x+1 = (x-1)(x^2-1) = (x-1)(x-1)(x+1)$$

$$\frac{4x}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x-1)^2(x+1)}$$

$$= \frac{x^2(A+B) + x(-2A+C) + (A-B+C)}{(x-1)^2(x+1)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ C-2A=4 \\ A-B+C=0 \end{cases} \sim \begin{cases} 2A+C=0 \\ C-2A=4 \end{cases} \sim \{ 2C=4$$

$$\Rightarrow C=2, \Rightarrow A=-1 \Rightarrow B=1.$$

$$\cdots \circledast = x^2/2 + x + C_0 + \int \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}\,dx$$

$$= x^2/2 + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C_1$$

§ Non Repeated Quadratic Factors

Suppose $Q(x) = ax^2+bx+c$ is irreducible. then

" $\dfrac{Ax+B}{ax^2+bx+C}$ " will appear in the partial frac expansion.

EXAMPLE

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \underset{\substack{\uparrow \\ \text{linear} \\ \text{non-repeated}}}{\frac{A}{x-2}} + \underset{\substack{\uparrow \\ \text{quad.} \\ \text{non-repeat}}}{\frac{Bx+C}{x^2+1}} + \underset{\substack{\uparrow \\ \text{quad} \\ \text{non-repeat.}}}{\frac{Dx+E}{x^2+4}}$$

EXAMPLE $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx = (*)$

$$\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow A(x^2+4) + (Bx+C)x = 2x^2-x+4 \xrightarrow{\text{hidden work}} (A,B,C) = (1,1,-1).$$

$$(*) = \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx = \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

§ Repeated Irreducible Quadratic Factors

Suppose $(ax^2+bx+c)^r$ is a factor of $Q(x)$ and ax^2+bx+c is irreducible, then

$$\frac{A_0x+B_0}{ax^2+bx+c} + \frac{A_1x+B_1}{(ax^2+bx+c)^2} + \cdots + \frac{A_{r-1}x\ B_{r-1}}{(ax^2+bx+c)^r}$$

will appear in the partial fraction decomp

EXAMPLE

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1}$$

$$+ \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}.$$

EXAMPLE: $\int \frac{1-x-2x^2-x^3}{x(x^2+1)^2}\,dx = *$

$$\frac{1-x-2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\Rightarrow 1-x-2x^2-x^3 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$\Rightarrow (A,B,C,D,E) = (1,-1,-1,1,0)$$

$$* = \int \frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \arctan x - \frac{1}{2}\frac{1}{x^2+1} + C$$

~~EXPAND~~

EXERCISES Expand the following via partial fraction expansion.

- $\dfrac{2x}{(x+3)(3x+1)}$
- $\dfrac{2x+1}{(x+1)^3(x^2+4)^2}$
- $\dfrac{x^3}{x^2+4x+3}$
- $\dfrac{x^4}{x^4-1}$

EXERCISE Evaluate

- $\displaystyle\int \frac{r^2}{r+4}\,dr$
- $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$
- $\displaystyle\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$
- $\displaystyle\int \frac{1}{x^3-1}\,dx$