a

IDENTITIES

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin mx \cdot \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\sin mx \cdot \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

$$\cos mx \cdot \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

§ TRIG INTEGRALS

Integrals of the form:

$$\int \sin^m x \cos^n x \, dx = (1).$$

$\boxed{\text{CASE: } m \text{ odd} \Rightarrow m = 2k+1}$

$$(1) = \int (\sin^2 x)^k \sin x \cos^n x \, dx$$

$$= \int \sin x \, (1-\cos^2 x)^k \cos^n x \, dx$$

$$u = \cos x \Rightarrow du = -\sin x \, dx$$

$$= -\int \underbrace{(1-u^2)^k u^n}_{\text{a polynomial in } u \text{ — always integral}} \, du$$

$\boxed{\text{CASE } n \text{ ~~even~~ odd} \Rightarrow n = 2l+1}$

$$(1) = \int \sin^m x \, (\cos^2 x)^l \cos x \, dx$$

$$= \int \sin^m x \, (1-\sin^2 x)^l \cos x \, dx$$

$$u = \sin x \rightarrow du = \cos x \, dx$$

$$= \int \underbrace{u^m (1-u)^l}_{\text{polynomial in } u \Rightarrow \text{integrable}} \, du$$

CASE m & n are even

$$① = \int (\sin^2 x)^k \cdot (\cos^2 x)^\ell \, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^k \cdot \left(\frac{1+\cos 2x}{2}\right)^\ell dx$$

double angle

$$= \frac{1}{2^{k+\ell}} \int (1-\cos 2x)^k \cdot \underbrace{(1+\cos 2x)^\ell}_{\text{lower degree.}} dx$$

EXAMPLE: $\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \sin x \cos^2 x \, dx$

$$= \int (1-\cos^2 x) \sin x \cos^2 x \, dx$$

$$u = \cos x \Rightarrow -du = \sin x \, dx$$

$$= -\int (1-u^2) u^2 \, du = -\int u^2 - u^4 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C.$$

$$= \boxed{\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C}$$

<u>EXAMPLE</u> $\int \cos^5 x\, dx = \int \cos x\, (\cos^2 x)^2\, dx$

$= \int \cos x\, (1-\sin^2 x)^2 dx$

$u = \sin x \Rightarrow du = \cos x\, dx$

$= \int (1-u^2)^2 du = \int 1 - 2u^2 + u^4\, du$

$= u^5/5 - 2u^3/3 + u + C$

$$= \boxed{\frac{\sin^5 x}{5} - \frac{2\sin^3 x}{5} + \sin x + C}$$

<u>EXAMPLE</u> $\int \sin^2 x\, \cos^4 x\, dx$

$$= \int \sin^2 x \cdot (\cos^2 x)^2\, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int 1 - \cos 2x - \cos^2 2x - \cos^3 2x\, dx$$

$$= \frac{1}{8}\left[x - \frac{1}{2}\sin 2x - \int \cos^2 2x + \cos^3 2x\, dx\right]$$

- $\int \cos^2 2x\, dx = \int \frac{1+\cos 4x}{2} dx = \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + C_0$
- $\int \cos^3 2x\, dx = \int \cos 2x\, \cos^2 2x\, dx$

$= \int \cos 2x\, (1-\sin^2 2x)\, dx$

$\therefore \int \cos 2x \,(1-\sin^2 2x)\,dx$

$u = \sin 2x \Rightarrow du = 2\cos 2x\,dx \Rightarrow \frac{1}{2}du = \cos 2x\,dx$

$= \frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}(u - u^3/3) + C_1$

$= \dfrac{\sin 2x}{2} - \dfrac{\sin^3 2x}{6}$

Thus: $\int \sin^2 x \cos^4 x\,dx$

$= \dfrac{x}{8} - \dfrac{\sin 2x}{16} - \dfrac{x}{16} - \dfrac{\sin 4x}{72} - \dfrac{\sin 2x}{16} - \dfrac{\sin^3 2x}{48} + C_2$

$= \frac{1}{16}\left[x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right] + C.$

EXAMPLE: $\int_0^{\pi/4} [1 + \cos 4x]^{\frac{1}{2}}\, dx =$

$$\cos^2 2x = \frac{1 + \cos 4x}{2} \Rightarrow 2\cos^2 2x = \cos 4x + 1$$

$$= \int_0^{\pi/4} [2\cos^2 2x]^{\frac{1}{2}}\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx$$

$$= \sqrt{2}\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \sqrt{2}\left[\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0\right] = \boxed{\sqrt{2}/2}$$

EXAMPLE: $\int \tan^4 x\, dx = \int \tan^2 x\,(\sec^2 x - 1)\, dx$

$$= \int \tan^2 x \sec^2 x - \tan^2 x\, dx$$

$$= \int \tan^2 x \cdot \sec^2 x + 1 - \sec^2 x\, dx$$

$$= x - \tan x + \int \tan^2 x \sec^2 x\, dx$$

$$u = \tan x \Rightarrow du = \sec^2 x\, dx$$

$$= x - \tan x + \int u^2\, du$$

$$= x - \tan x + u^3/3 + C$$

$$= \boxed{x - \tan x + \frac{1}{3}\tan^3 x + C.}$$

EXAMPLE $\int \sec^3\theta \, d\theta$ by IBP $= (*)$

$u = \sec\theta \qquad V = \int \sec^2\theta \, d\theta = \tan\theta$

$du = \sec\theta \tan\theta \, d\theta \qquad dV = \sec^2\theta \, d\theta$

$(*) = \sec\theta \tan\theta - \int \tan^2\theta \sec\theta \, d\theta$

$= \sec\theta \tan\theta - \int \sec^3\theta - \sec\theta \, d\theta$

$\Rightarrow 2\int \sec^3\theta \, d\theta = \sec\theta \tan\theta + \int \sec\theta \, d\theta$

$= \sec\theta \tan\theta + \ln\underbrace{|\sec\theta + \tan\theta|}_{\text{by table lookup}} + c$

$\Rightarrow \int \sec^3\theta \, d\theta = \frac{1}{2}\left[\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| + c\right]$

EXAMPLE: $\int \tan^4 x \sec^4 x \, dx$

$$= \int \tan^4 x \, (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int (\tan^4 x + \tan^6 x) \sec^2 x \, dx$$

$$u = \tan x \Rightarrow du = \sec^2 x \, dx$$

$$= \int u^4 + u^6 \, du = u^5/5 + u^7/7 + C$$

$$= \boxed{\frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C}$$

EXAMPLE: $\int \sin 3x \cos 5x \, dx$

$$= \int \frac{1}{2}\left[\sin(-2x) + \sin(8x)\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\cos(-2x) - \frac{1}{8}\cos(8x) + C\right]$$

$$= \frac{1}{4}\cos(-2x) - \frac{1}{8}\cos(8x) + C/2$$

EXERCISE:

- $\int 7\cos^7 t \, dt$

- $\int 8\cos^3 2\theta \sin 2\theta \, d\theta$

- $\int \sec^4\theta \tan^2\theta \, d\theta$

- $\int \sin^3\theta \cos^3\theta \, d\theta$