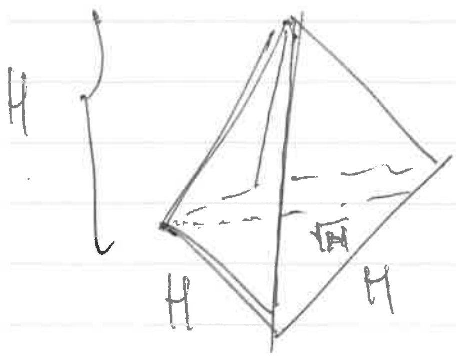
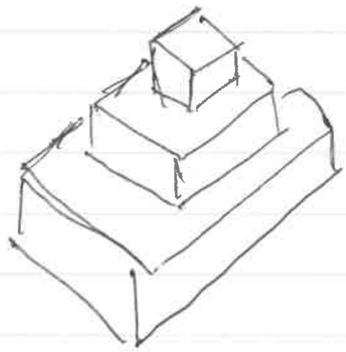
(1)

§ Volumes by Cross-Sections

<u>MOTIVATION</u> What is the volume of the "equilateral square-based pyramid" of height ~~...~~ H?



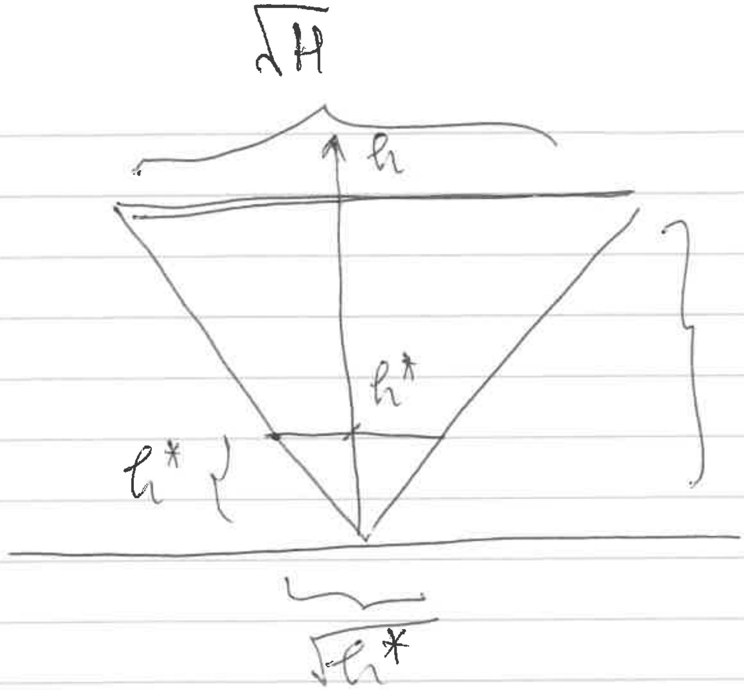
is approximated by

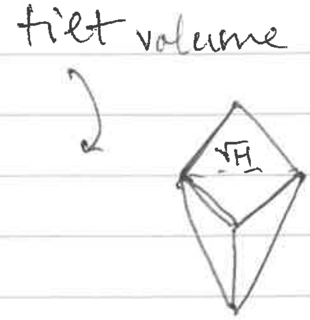


We need the base of the prism at various sample heights.

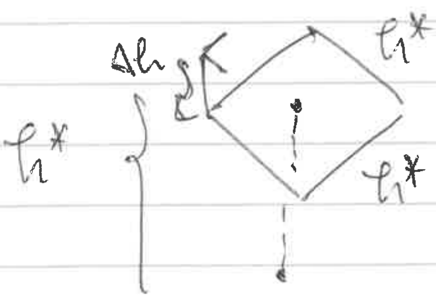
Thankfully an "equilateral square based pyramid" has the following property.

If we slice the pyramid through the bases diagonal and look at the cross section, we get.





So at height $h^* \in [0, H]$ we have



at height h^*.

$$\text{Volume} \approx \sum_{k=1}^{N} \text{volume of prism @ } h^*_k$$

$$\approx \sum_{k=1}^{N} l \times w \times h \text{ @ } h^*_k$$

$$\approx \sum_{k=1}^{N} h^*_k \, h^*_k \, \cancel{h^*_k} \, \Delta h$$

taking $N \to \infty$

$$\lim_{N\to\infty} \sum_{k=1}^{N} (h_k^*)^2 \Delta h = \int_0^H h^2\, dh = \frac{h^3}{3}\Big|_0^H$$

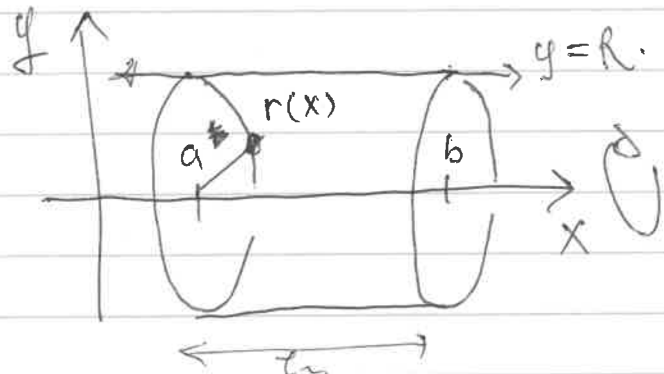
$$* \Delta h = \frac{H-0}{N} \qquad\qquad = \frac{H^3}{3}.$$

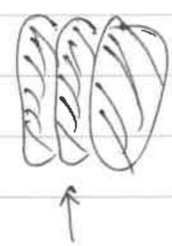
Def$^{\underline{n}}$ The volume of a solid whose cross-sections, $A(x)$, are known over $x \in [a,b]$ is given by

$$V = \int_a^b A(x)\, dx.$$

§ Volumes of Rotations

Volume (Cylinder)





summing discs.

Cross-section is easy; its always

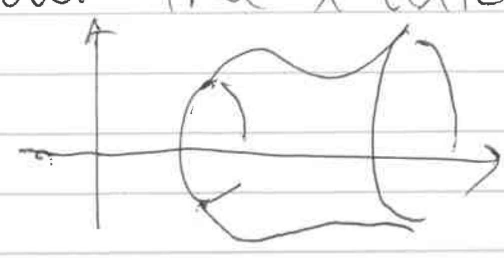
$$r(x) = R \quad \text{for } x \in [a,b].$$

$$V = \int_a^b \pi R^2 \, dx = \cancel{\pi R^3/3 \Big|_a^b = \pi}$$

$$= \pi R^2 x \Big|_a^b = \pi R^2 (b-a)$$

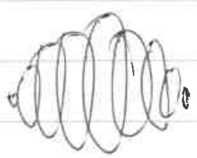
when just given height $= \pi R^2 h$

What about other curves? And their rotations about the x-axis?

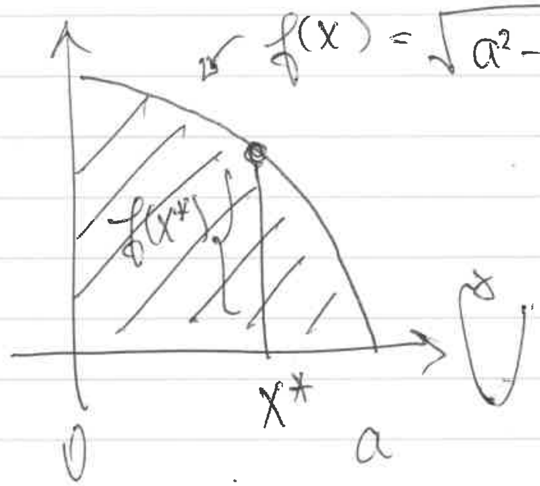


Volume (Sphere)

Rotate the circle $x^2 + y^2 = a^2$ about the x-axis to obtain the volume of the sphere of radius a.



It's simpler to do $\frac{1}{2}$ Vol (sphere)



Radius at x^*
$= r(x) = (a^2 - x^2)^{\frac{1}{2}}$

$$\frac{1}{2}\text{Vol (sphere)} = \int_0^a \pi\, r(x)^2\, dx = \pi \int_0^a a^2 - x^2\, dx$$

$$= \pi \left[a^2 x - x^3/3 \right]_0^a = \pi \left[a^3 - a^3/3 \right]$$

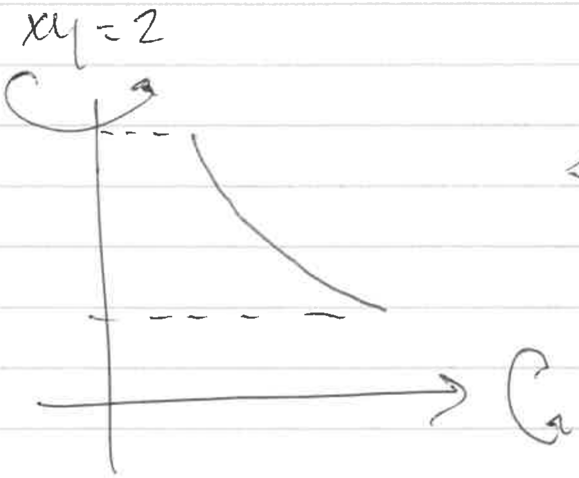
$$= \frac{2}{3}\pi a^3 \implies$$ Vol of sphere of radius a is

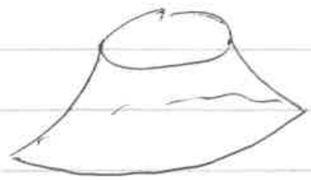
$$4/3\, \pi a^3.$$

Note You can also spin along y-axis

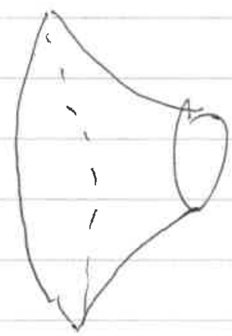
EXERCISE Find the ~~area~~ volume obtained by rotating the curve $xy=2$ over $y=1..4$ about

- y-axis
- x-axis





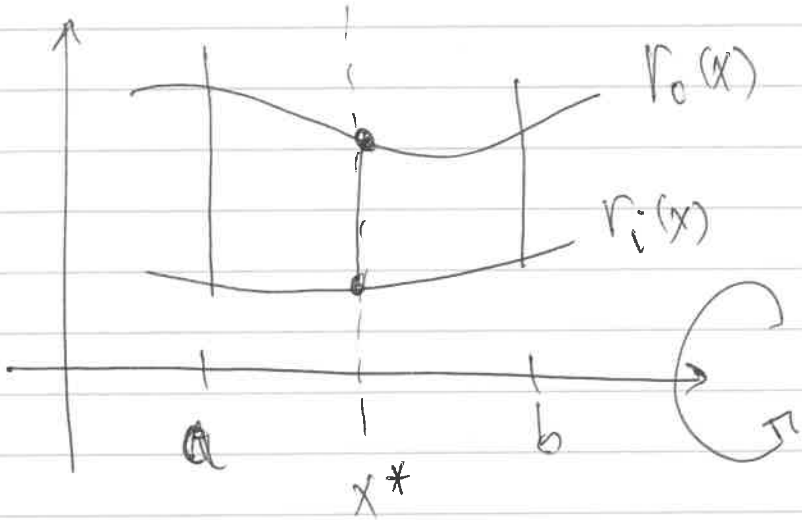
about y



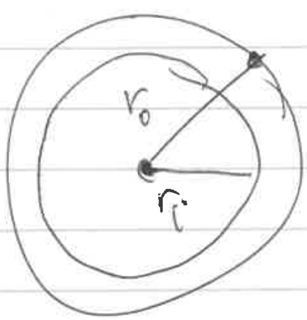
about x

Spinning

§ Spinning Bounded Regions



Geometry: Here we are summing "washers"



$$\text{Area(Washer)} = \pi r_o^2 - \pi r_i^2 = \pi(r_o^2 - r_i^2)$$

$$\Rightarrow V = \int_a^b \pi\left(r_o(x)^2 - r_i(x)^2\right)\,dx$$

EXERCISE $y = x^2+1$ and $y = 3-x$ intersect at $(1,2)$ & $(-2,5)$. and form a bounded region.

What volume is produced by rotating this region (a) about x-axis

(b) about y-axis

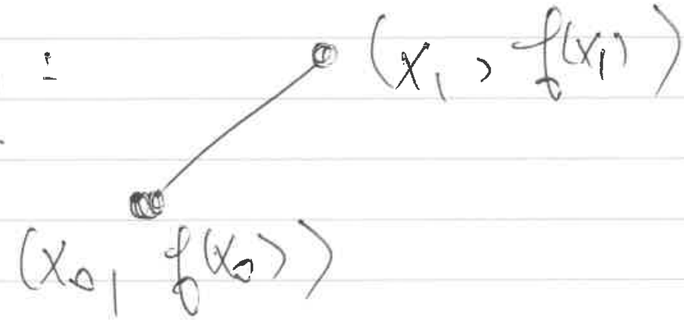
§ Arc length

Motivation: Suppose you lay a rope along a curve f over $[a,b]$ — how long is it?

Def$^{\underline{n}}$ Chord

Given a function f, a chord is a line segment connecting two points on the curve f.

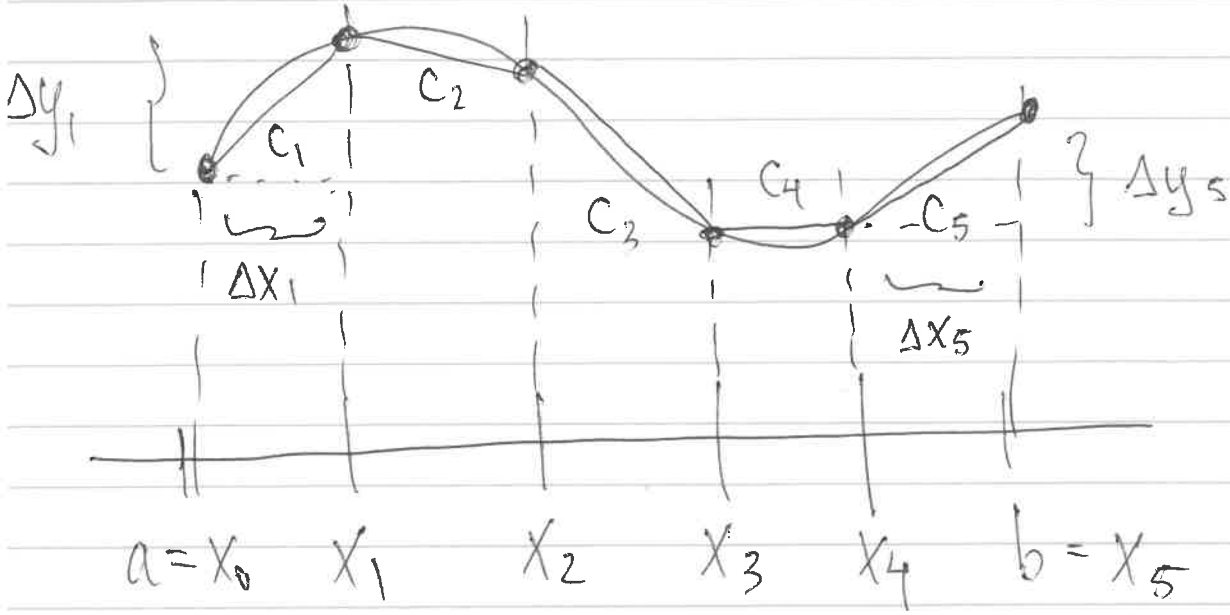
EXAMPLE:



is a cord

w/ length $\left((x_1 - x_0)^2 + (f(x_1) - f(x_0))^2 \right)^{\frac{1}{2}}$

Break curve into chords:



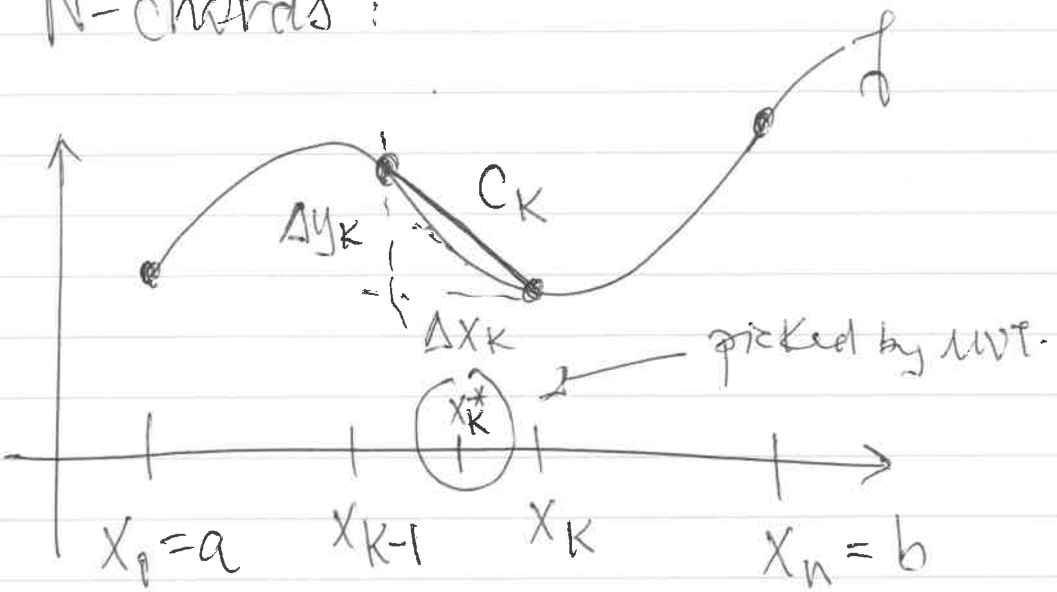
$$\text{ARC LENGTH} \cong \sum_{K=1}^{5} \text{length of cord } K$$

~~letting $\Delta x_K = x_{K+1} - x_K$ $\Delta y_K = f(x_{K+1}) - f(x_K)$~~

letting $\Delta x_K = x_K - x_{K-1}$ & $\Delta y_K = f(x_K) - f(x_{K-1})$

$$\cong \sum_{K=1}^{5} \left(\Delta x_K^2 + \Delta y_K^2\right)^{\frac{1}{2}}$$

For N-chords:



$\Delta x_K = x_K - x_{K-1} \qquad \Delta y_K = f(x_K) - f(x_{K-1})$

length of $C_K = \|C_K\| = \left[\Delta x_K^2 + \Delta y_K^2\right]^{\frac{1}{2}} = (*)$

NOTICE! (Magic Step)

MVT $\Rightarrow \exists x_K^* \in [x_{K-1}, x_K] : f'(x_K^*) = \dfrac{\Delta y_K}{\Delta x_K}$

$\Downarrow$

$(*) = \left[\Delta x_K^2 + f'(x_K^*)^2 \Delta x_K^2\right]^{\frac{1}{2}} \qquad \Delta y_K = f'(x_K^*)\Delta x_K$

$$\text{"AL"} = \lim_{N\to\infty} \sum_{K=1}^{\infty} \left[\Delta x_K^2 + f'(x_K^*)^2 \Delta x_K^2\right]^{\frac{1}{2}}$$

$$= \lim_{N \to \infty} \sum_{K=1}^{\infty} \left[1 + f'(x_K^*) \right]^{\frac{1}{2}} \Delta x$$

Notice $\Delta x \to 0$ as $N \to \infty$ and thus $x_K^* = x_K$ in this limit.

$$= \lim_{N \to \infty} \sum_{K=1}^{\infty} \left[1 + f'(x_K)^2 \right]^{\frac{1}{2}} \Delta x \qquad \Delta x = \frac{b-a}{N}$$

$$= \int_a^b \left[1 + f'(x)^2 \right]^{\frac{1}{2}} dx$$

$$\text{OR} \quad = \int_a^b \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx$$

Def$^{\underline{n}}$ Arc-Length

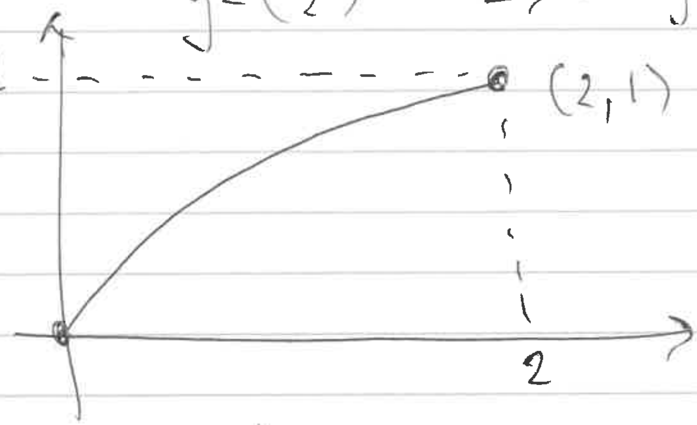
The "arc-length" of a curve f ($y = f(x)$) for $x \in [a,b]$ is given by

$$\int_a^b \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx.$$

provided $f(x)$ is diffable over $[a,b]$

EXAMPLE What is the arc-length of $y=\left(\frac{x}{2}\right)^{2/3}$ for $x\in[0,2]$.

$$y=\left(\frac{x}{2}\right)^{2/3} \Rightarrow x=2y^{3/2}$$



Notice $\frac{dy}{dx}=\frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\cdot\frac{1}{2}$ is not defined at $x=0$.

But $\frac{dx}{dy}=3y^{\frac{1}{2}}$ is.

So we give the arc-length by:

$$AL=\int_0^1\left(1+(3\sqrt{y})^2\right)^{\frac{1}{2}}dy$$

$$=\int_0^1(1+9y)^{\frac{1}{2}}dy = (*) = \frac{2}{27}\left[10\sqrt{10}-1\right]$$

$$(*) = \int_0^1 [1 + 9y]^{\frac{1}{2}}\,dy = \frac{1}{9}\int_1^{10} u^{\frac{1}{2}}\,du = (**)$$

$u = 1 + 9y \Rightarrow du = 9dy \Rightarrow \frac{1}{9}du = dy$

$y = 0..1 \Rightarrow u = 1..10$

$$(**) = \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{2}{27}\left[10\sqrt{10} - 1\right] \quad \boxtimes$$

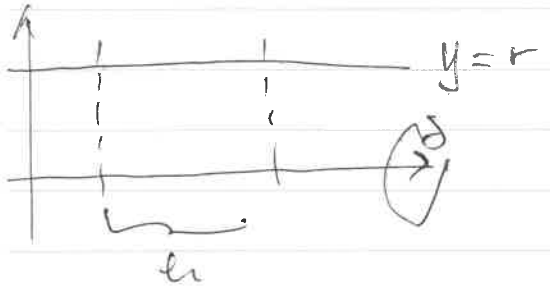
<u>EXERCISE</u>: Find arc-length of

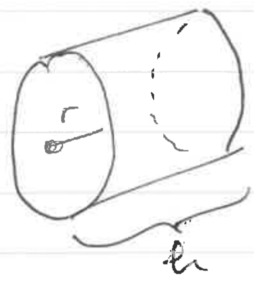
$$y = \frac{1}{3}(x^2 + 2)^{3/2} \quad \text{for } x \in [0, 3].$$

LECTURE

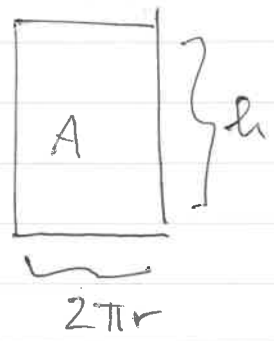
§6.4 Surface Area:

SA(cylinder) $= 2\pi r \cdot h$.





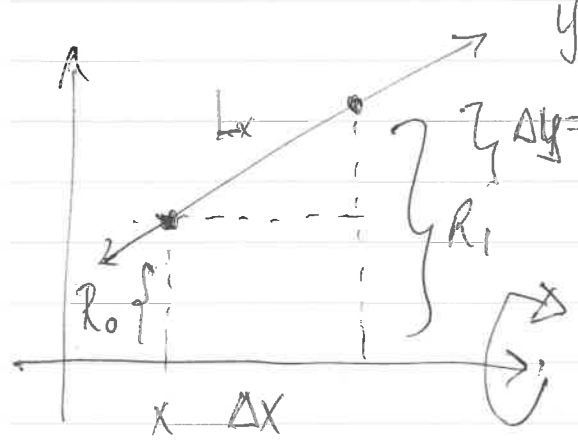
Cut and unrolled ...

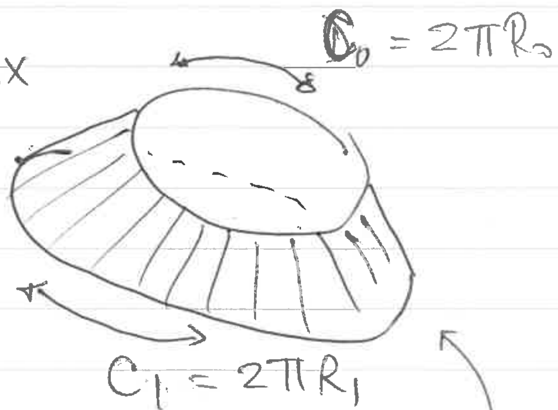


$A = 2\pi r \cdot h$

SA(FRUSTUM)

$y = mx + b \ : \ m, b \in \mathbb{R}$



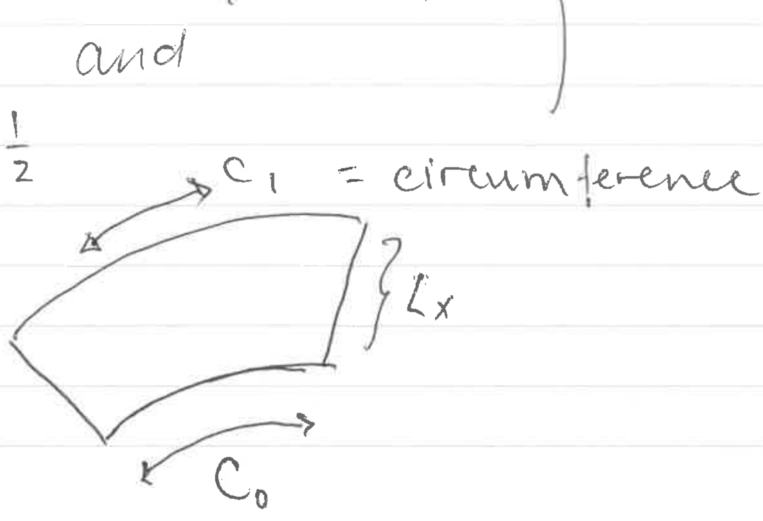


Notice: $R_1 = R_0 + f'(x)\Delta x$ and

$$L_x = \left[\Delta x^2 + f'(x)^2 \Delta x^2\right]^{\frac{1}{2}}$$

Cut and unravelled

(*) looks like:



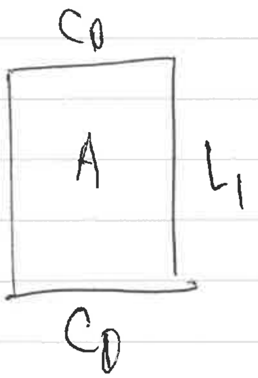
But shrinking Δx we see

$$\cancel{\lim_{\Delta x \to 0} C_1 = 2\pi R}$$

$$\lim_{\Delta x \to 0} C_1 = \lim_{\Delta x \to 0} 2\pi (R_0 + f'(x)\Delta x)$$

$$= 2\pi R_0 = C_0$$

So when $\Delta x \to 0$ the frustum is



$A = 2\pi R_0 L_1$.

Thus

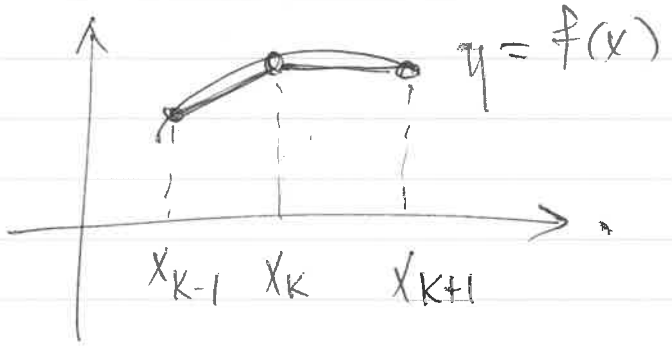
$$\lim_{\Delta x \to 0} SA(\text{frustum}) = SA(\text{Cylinder})$$

$$= 2\pi R_0 L x$$

$$= 2\pi x \left[\Delta x^2 + f'(x)^2 \Delta x^2\right]^{\frac{1}{2}}$$

$$= 2\pi x \left[1 + f'(x)^2\right]^{\frac{1}{2}}$$

The surface area obtained by rotating an arbitrary curve $y = f(x)$ is given by adding fostums.



Defn SA obtained by rotating $y = f(x)$ $x = a..b$ about the x-axis is given by

$$SA = \int_a^b 2\pi f(x)\left[1 + f'(x)^2\right]^{\frac{1}{2}} dx.$$

OR

$$SA = \int_a^b 2\pi y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx \qquad (*)$$

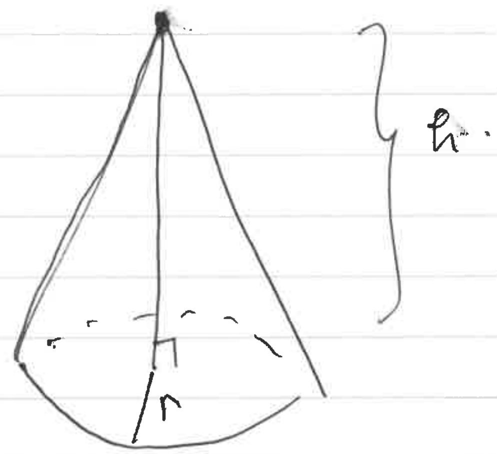
Rotations about y-axis is obtained by swapping $y \leftrightarrow x$ in $(*)$ and computing new bounds a, b along the y-axis.

iClicker.

Do: <u>Example 1</u> $y = 2\sqrt{x}$, $x = 1..2$ about x-axis.

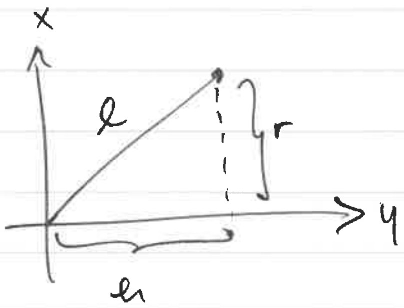
<u>Question 17</u> $x = \frac{1}{3}y^3$ $y = 0..1$ about y-axis.

<u>Assesment</u> Compute the surface area of the ~~cylinder~~ CONE (no lid).



§6.4 Surface Area of a Cone (no lid):

$y = \frac{r}{h} \cdot x$



$$y = \frac{r}{h} x \implies \frac{dy}{dx} = \frac{r}{h} \implies \left(\frac{dy}{dx}\right)^2 = \left(\frac{r}{h}\right)^2$$

$$SA = \int_0^h 2\pi f(x) \left[1 + \left(\frac{dy}{dx}\right)\right]^{\frac{1}{2}} dx$$

$$= 2\pi \int_0^h \frac{r}{h} x \left[1 + \frac{r^2}{h^2}\right]^{\frac{1}{2}} dx$$

$$\int_0^h x \, dx = \frac{x^2}{2}\Big|_0^h = \frac{h^2}{2}$$

$$= 2\pi \frac{r}{h} \left[1 + \frac{r^2}{h^2}\right]^{\frac{1}{2}} \int_0^h x \, dx$$

$$= 2\pi r/h \left[1 + \frac{r^2}{h^2}\right]^{\frac{1}{2}} \frac{h^2}{2}$$

$$= \pi r \left[h^2 + r^2\right]^{\frac{1}{2}} \quad \dots = \pi r l$$

§6.4

EXAMPLE I:

Rotate $y = 2\sqrt{x}$ $x = 1..2$ about the x-axis.

$$\frac{dy}{dx} = 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{x}$$

$$SA = \int_1^2 2\pi \, 2\sqrt{x} \left[1 + \frac{1}{x}\right]^{\frac{1}{2}} dx$$

$$= 4\pi \int_1^2 [x+1]^{\frac{1}{2}} dx$$

$u = x+1 \Rightarrow du = dx$
$x = 1..2 \Rightarrow u = 2..3$

$$= 4\pi \int_2^3 u^{\frac{1}{2}} du = 4\pi \left[\tfrac{2}{3} u^{3/2}\right]_2^3$$

$$= \tfrac{8}{3}\pi \left[3^{3/2} - 2^{3/2}\right] = \tfrac{8}{3}\pi \left[3\sqrt{3} - 2\sqrt{2}\right]$$

Q17

§6.4 $\cancel{Q17}$ $\quad y=(3x)^{\frac{1}{3}} \Rightarrow y^3 = 3x \Rightarrow x = \frac{1}{3}y^3$

Rotate about y-axis, $\quad y = 0..1$

$$\frac{dx}{dy} = y^2 \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4$$

$$SA = \int_0^1 2\pi \frac{1}{3} y^3 \left[1+y^4\right]^{\frac{1}{2}} dy$$

$$\left\{= \int_0^1 2\pi\, x(y) \left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dy\right\}$$

$$= \frac{2}{3}\pi \int_0^1 y^3 \left[1+y^4\right]^{\frac{1}{2}} dy$$

$u = 1+y^4 \rightarrow du = 4y^3\,dy \rightarrow \frac{1}{4}du = y^3\,dy$

$y = 0..1 \rightarrow u = 1..2$

$$= 2/3\,\pi \int_1^2 u^{\frac{1}{2}} \frac{1}{4} du = \frac{\pi}{6}\int_0^2 u^{\frac{1}{2}} du = \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]_0^2$$

$$= \left[2\sqrt{2}-1\right]\pi/9$$

∎