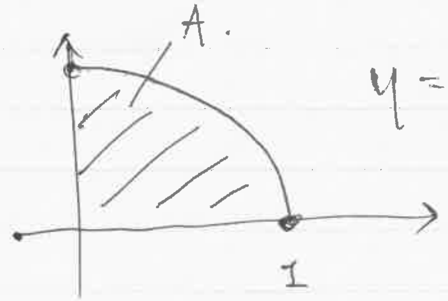
§ Areas / Estimating w/ Finite Sums

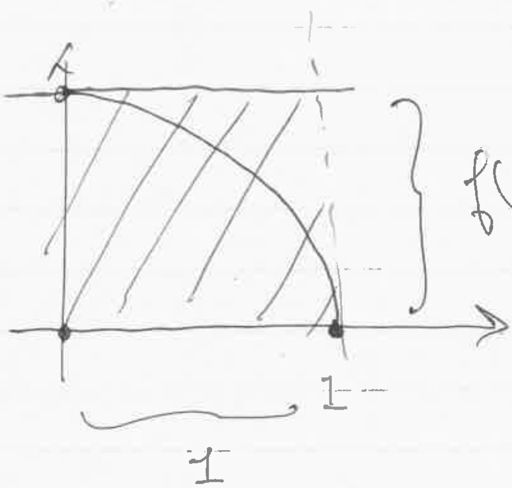
MOTIVATION: How can we find the area of "weird" shapes that do not have an area formula?

EXAMPLE Find the shaded area

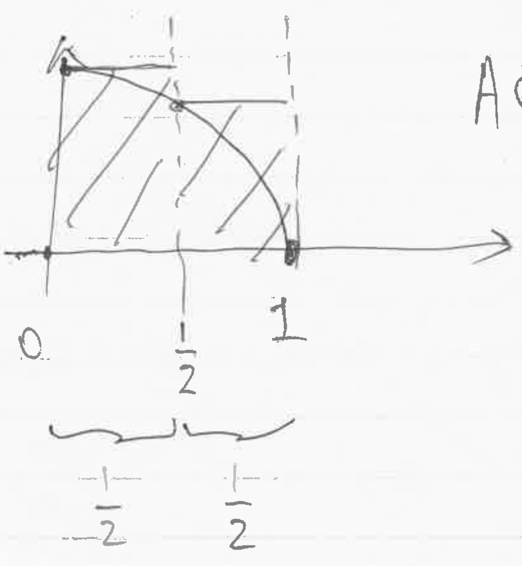


$y = 1 - x^2$

We can certainly bound (give lower and upper constraints) the area A.



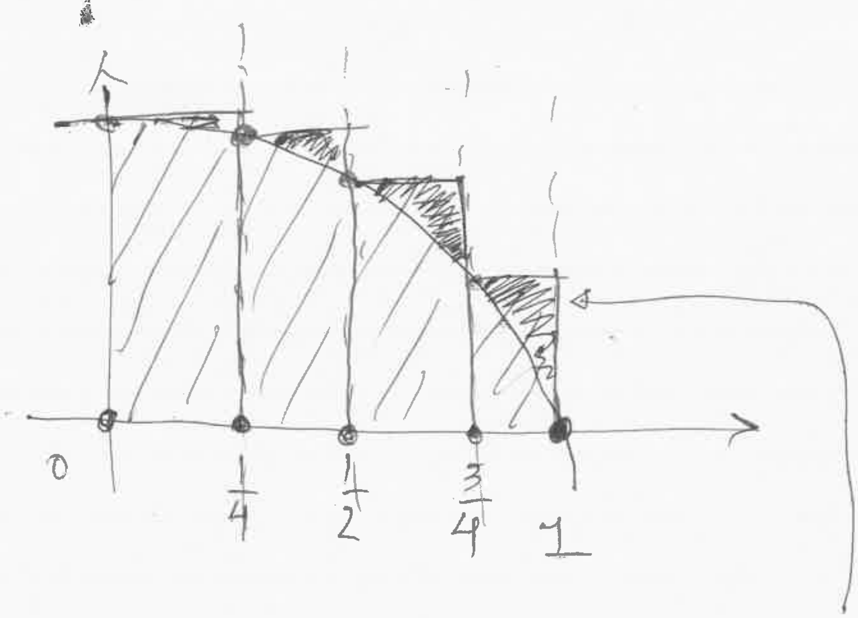
$A \in [0, 1]$



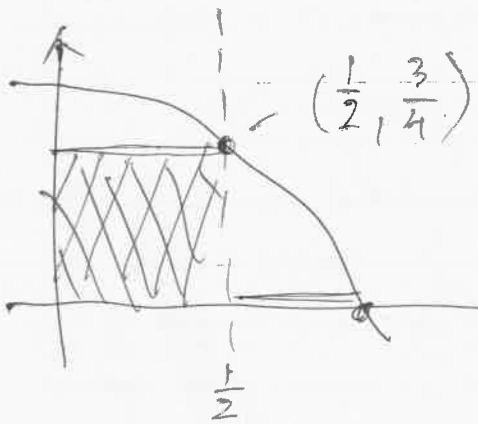
$$A \in \left[0, \tfrac{1}{2} \cdot f(0) + \tfrac{1}{2} f(\tfrac{1}{2})\right]$$
$$= \left[0, \tfrac{1}{2}(1 + 3/4)\right]$$
$$= [0, 7/8]$$

narrower, thus better, interval.

(*)

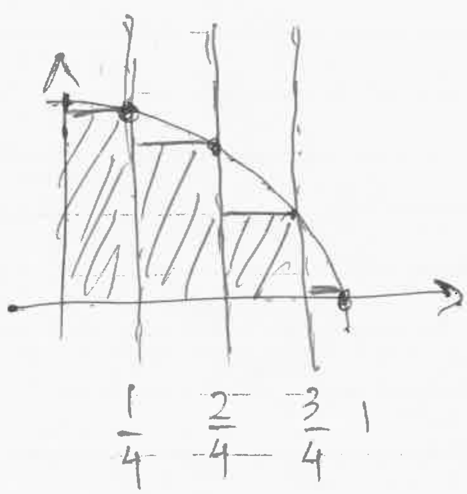


Notice: in each case we have <u>overestimated</u> the area.
We can <u>underestimate</u> as well.



$$A \in \left[\tfrac{1}{2} \cdot f(\tfrac{1}{2}) + \tfrac{1}{2} f(1),\ 25/32\right]$$
$$= \left[3/8,\ 25/32\right]$$

(**)



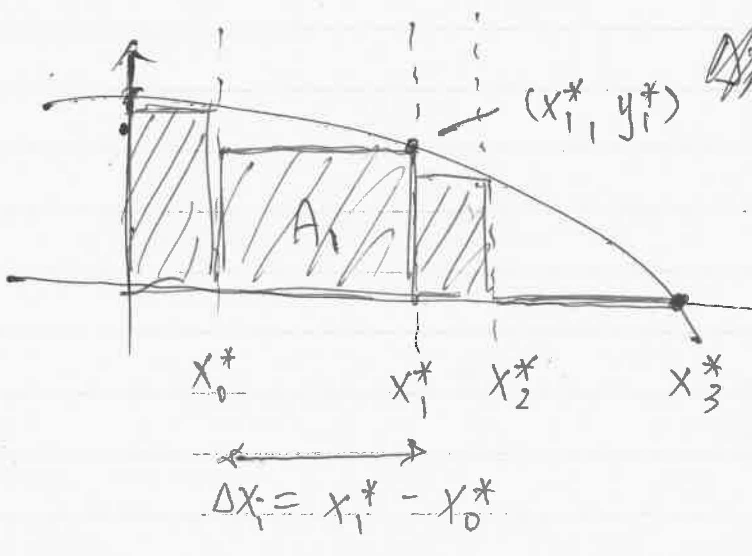
$$A \in \left[\tfrac{1}{4}\left(f(\tfrac{1}{4}) + f(\tfrac{2}{4}) + f(\tfrac{3}{4}) + f(1)\right),\ 25/32\right]$$
$$= \left[17/32,\ 25/32\right]$$

The idea is that w/ more rectangles our interval shrinks in width and our estimation of the area becomes more accurate.

§ Endpoints and Sampling

In order to form an approximation using rectangles we need to choose where (on the x-axis) to form the heights. These are called "samples" or "endpoints" denoted by "x_i^*" – the i^{th} sample.

Note: We have been equally spacing our samples but we can choose them randomly.

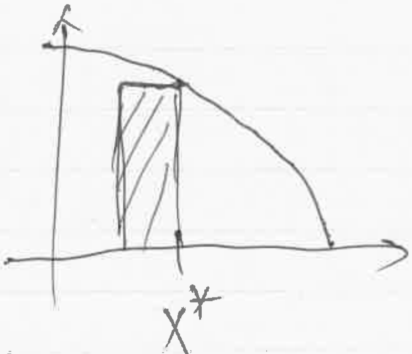


$$\Delta x_i = x_{i+1}^* - x_i^*$$

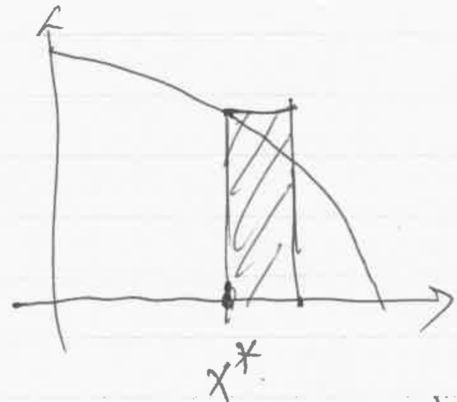
$$\Rightarrow A_1 = \Delta x_1 \cdot f(x_1^*)$$

We typically space the samples uniformally (i.e. evenly) along the interval.

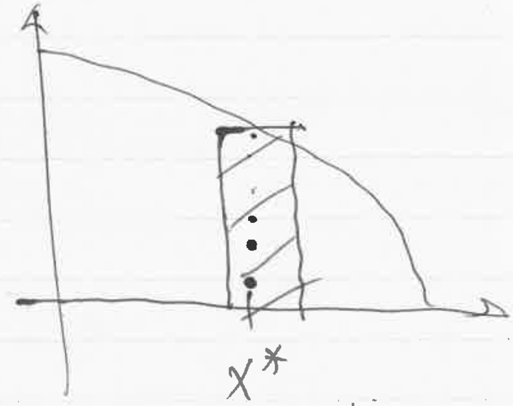
There are standard "endpoints" of an interval we use to form the rectangles height.



Right endpoint

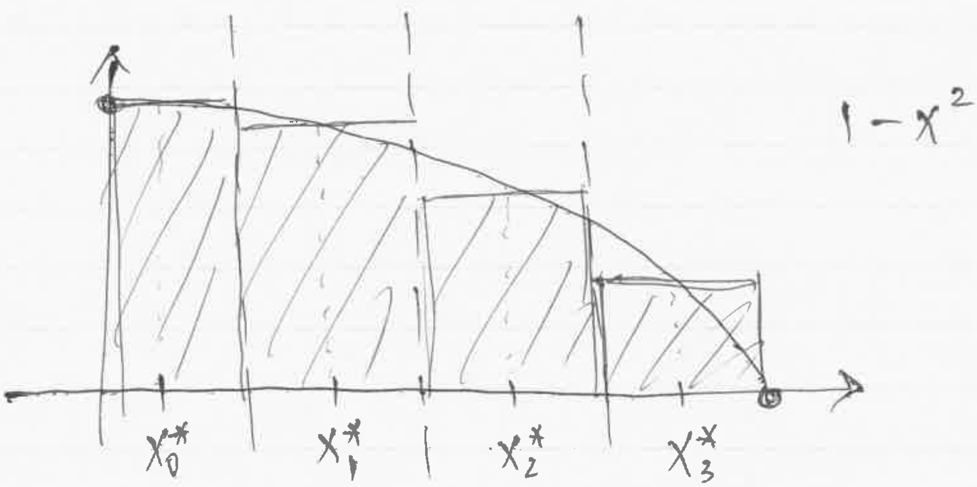


Left endpoint



Midpoint.

(*) and (**) are the four-rectangle approx. of the area under $1-x^2$ from $x=0..1$ using LE and RE resp. Now do middle.



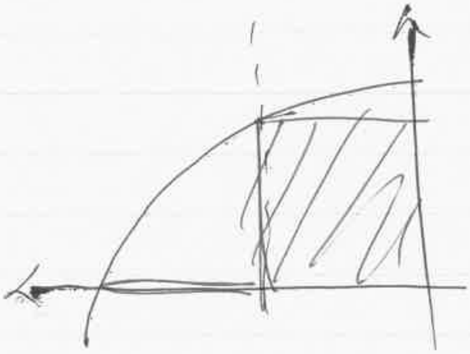
Note: The midpoint is the average of the LE/RE.

$$A \approx \frac{1}{4} f\left(\frac{1}{8}\right) + \frac{1}{4} f\left(\frac{3}{8}\right) + \frac{1}{4} f\left(\frac{5}{8}\right) + \frac{1}{4} f\left(7/8\right)$$

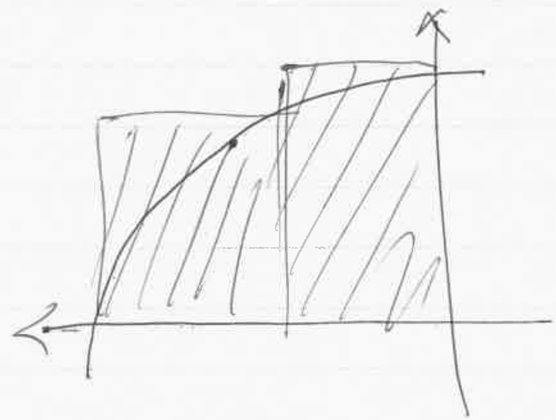
$$= \ldots = 43/64.$$

Notice over/under estimations change based on curvature of the function and are not only governed by LE/RE.

EXAMPLE: Consider area under $y = 1 - x^2$ for $x = -1 .. 0$.



LE - underestimate



RE - overestimate

EXERCISE Using two rectangles form approximations for area under the curve for:

1. $f(x) = x^2$, $x = 0..1$
2. $f(x) = x^3$, $x = 0..2$
3. $f(x) = \frac{1}{x}$, $x = 1..5$

Indicate if your estimate is over or under.

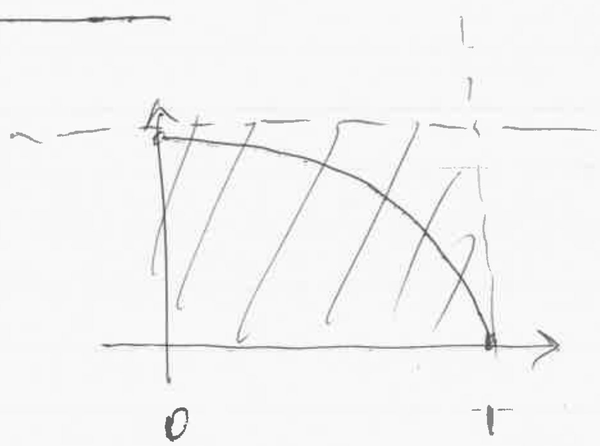
§ Average Values

What is the average value of a function $f(x)$ over $[a,b]$?

Remember: $\text{Average}(\{x_0, x_1, \ldots, x_n\})$
$$= \frac{x_0 + x_1 + \cdots + x_n}{n+1}.$$

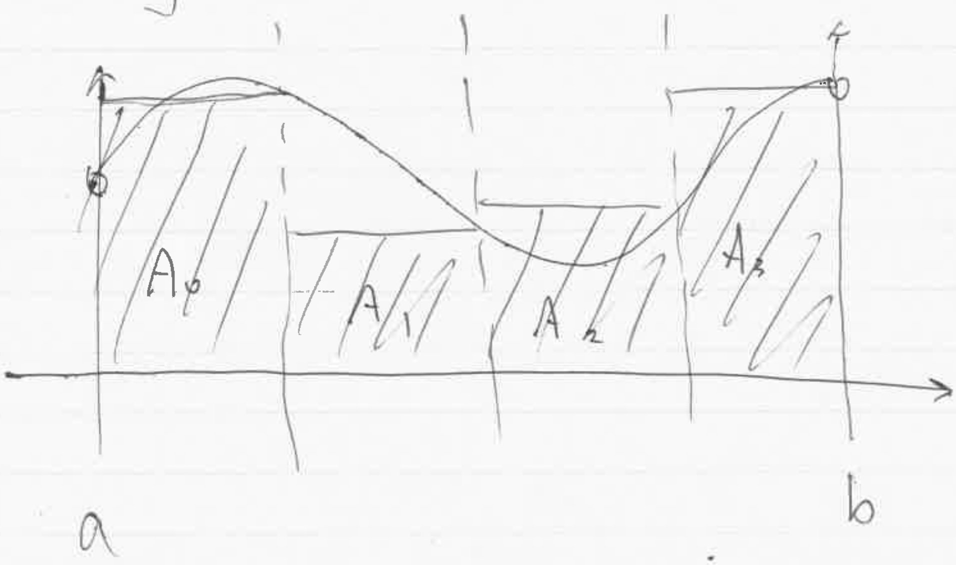
Geometrically, a constant function $f(x) = c$ has average value c over any interval.

EXAMPLE



$\text{Ave}(1 - x^2) < 1$
for unit interval.

Generally



$$\text{Ave}(f) \approx \frac{A_0 + A_1 + A_2 + A_3}{4}$$ and the approx will improve w/ more rectangles.

$\int$ Physics

Suppose we wanted to work out how far a car has travelled using only instantaneous velocity information — this ends up being the area under the velocity curve.

Recall: Dist = Velocity · Time

EXAMPLE Sampling (ie. looking at the spedometer) every five minutes produces Table 1 - Estimate the distance travelled.

TABLE 1

min	velocity m/s
0	1
5	1.2
10	1.7
15	2.0
20	1.8
25	1.6
30	1.4

Distance Travelled

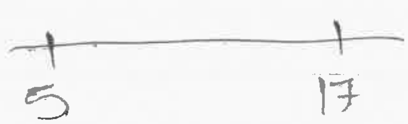
Interval (s)	LE	RE
0 - 5	5.60.(1)	5.60.(1.2)
5 - 10	5.60.1.2	5.60.(1.7)
10 - 15	⋮	⋮
15 - 20	⋮	⋮
20 - 25	⋮	⋮
30 - 35	5.60 (1.6)	5.60 (1.4)
total:		

EXERCISE You've driving a twisty road. You know the inst. velocity every 10s. Estimate the length of the road by giving an interval where the exact value must lie

time (s)	0	10	20	30	40	50
velocity (ft/sec)	0	44	15	35	30	44

EXERCISE Using critical points, can you estimate the area under $f(x) = (x-1)(x-2)(x-3)$ from $x = 0..5$ that is gauranteed an over-estimate.

EXERCISE Suppose we want to break the interval



into 4, 8, N ... equal parts. What are the LE/RE/M?

§ Sigma Notation

Let $A = \{1, 2, 3, 4\}$

$$\Rightarrow \sum_{a \in A} a = 1 + 2 + 3 + 4$$

$$\Rightarrow \sum_{a \in A} a^2 = 1^2 + 2^2 + 3^2 + 4^2$$

$$\Rightarrow \sum_{a \in A} f(a) = f(1) + f(2) + f(3) + f(4)$$

We can also do

$$\sum_{K=1}^{4} K = 1 + 2 + 3 + 4$$

$$\sum_{K=1}^{4} = f(1) + f(2) + f(3) + f(4)$$

K "counts" by integers.