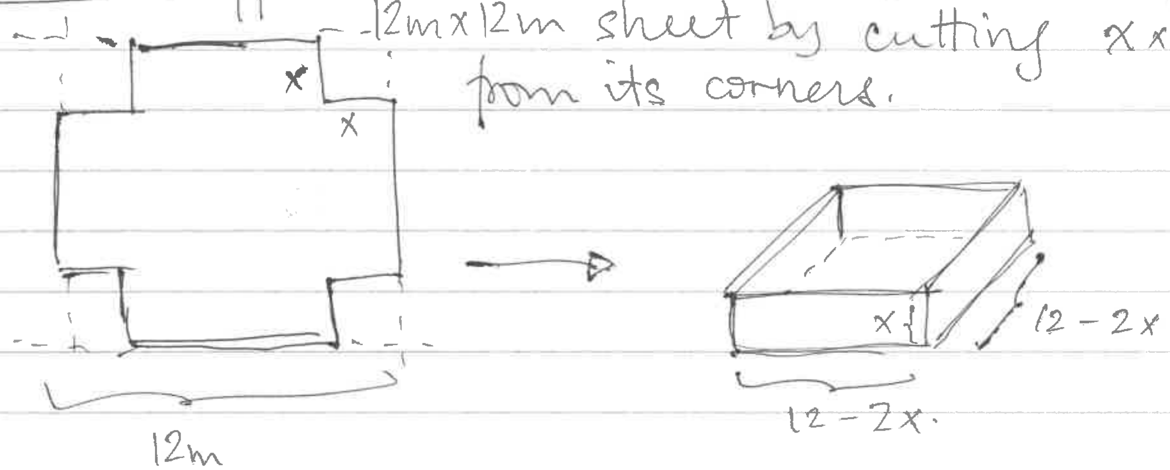
§Applied Optimization.

Use calculus to maximize or minimize some desired property.

EXAMPLE Suppose we make a box w/ no lid from a 12m x 12m sheet by cutting x x x squares from its corners.



Volume: $V = (12-2x)(12-2x)(x) = 144x - 48x^2 + 4x^3$.

$\Rightarrow V' = 12x^2 - 96x + 144$

$V' = 0 \Rightarrow 0 = 12x^2 - 96x + 144 = 12(x-6)(x-2)$

V has critical points $(6,0)$ and $(2,128)$

Note $\operatorname{dom} V = [0,6]$ and $V(0)=0$, $V(6)=0$ — endpoints are minimums.

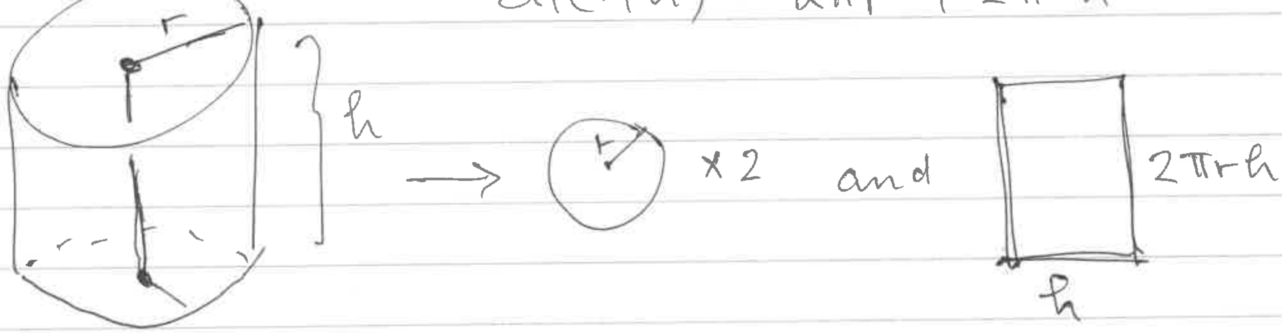
Max volume occurs at $(2,128)$.

$\Rightarrow$ max volume $= 128$.

<u>EXAMPLE</u> Design a one liter right-circular cylinder using the least material.

"material" means surface area.

$$SA(r,h) = 2\pi r^2 + 2\pi r h$$



Minimize SA under the constraint $Vol = V(r,h) = 1$.

minimize: (i.e. "optimize")

$$\begin{cases} SA(r,h) = 2\pi r^2 + 2\pi r h \\ V(r,h) = 1 = \pi r^2 h \end{cases}$$

Eliminate one variable from SA.

$$1 = \pi r^2 h \Rightarrow \frac{1}{r} = \pi r h \rightarrow SA = 2\pi r^2 + 2\cdot\frac{1}{r} : r \in (0,\infty)$$

$$\Rightarrow SA' = 4\pi r - 2/r^2$$

$$SA' = 0 \Rightarrow \frac{2}{r^2} = 4\pi r \Rightarrow r^3 = \frac{1}{2\pi} \Rightarrow r = \left(\frac{1}{2\pi}\right)^{\frac{1}{3}}$$

are critical points

Endpoints: $SA\big|_{r=0} = 2\pi 0^2 + 2 \cdot \text{“}\frac{1}{0}\text{”}$ undef...

$$\lim_{r \to 0^+} SA = \lim_{r \to 0^+} 2\pi r + 2 \cdot \frac{1}{r} = +\infty$$

$$\lim_{r \to \infty} SA = \text{“}\infty + \infty\text{”} = \infty.$$

minimum definitely <u>not</u> at endpoints

Critical points on SA

$$p = \left(\sqrt[3]{\frac{1}{2\pi}} \;,\; \underbrace{2\pi \left(\frac{1}{2\pi}\right)^{2/3} + 2\left(\frac{1}{2\pi}\right)^{-\frac{1}{3}}}_{\text{positive and } \underline{\text{finite}}} \right)$$

<u>Note</u> $SA'' = 4\pi + \frac{4}{r^3}$

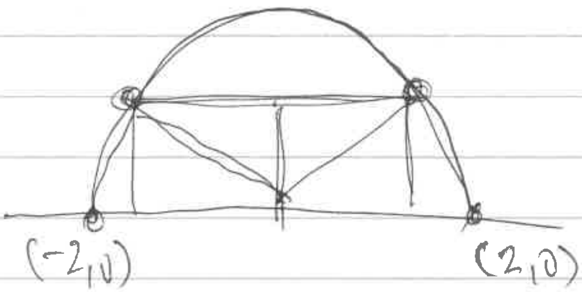
$$\Rightarrow SA''\left(\left(\frac{1}{2\pi}\right)^{\frac{1}{3}} \right) = 4\pi - \frac{4}{\frac{1}{2\pi}} = 4\pi + 8\pi \quad \cancel{4\pi}$$
$$= 12\pi > 0.$$

$\Rightarrow$ Upward inflection at p $\Rightarrow$ local & global min.

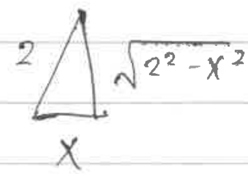
Minimum

$$SA = 2\pi\left(\frac{1}{2\pi}\right)^{2/3} + 2\left(\frac{1}{2\pi}\right)^{-\frac{1}{3}} = 2\pi\left(\frac{1}{2\pi}\right)^{2/3} + 2\cdot(2\pi)^{\frac{1}{3}}$$

EXERCISE: Inscribe a rectangle R in a semi-circle of radius 2. What is R's maximum area?



$A(x) = 4$ triangles



$x \in [0,2]$

$= 4 \cdot \frac{1}{2} \cdot x\,(4-x^2)^{\frac{1}{2}}$

$= 2x\,(4-x^2)^{\frac{1}{2}}$

$= 2x\sqrt{(2-x)(2+x)}$

Boundary: $A(0)=0$, $A(2)=0$ — not maximum.

$A'(x) = 2(4-x^2)^{\frac{1}{2}} + x(4-x^2)^{-\frac{1}{2}}(-2x)$ $(*)$

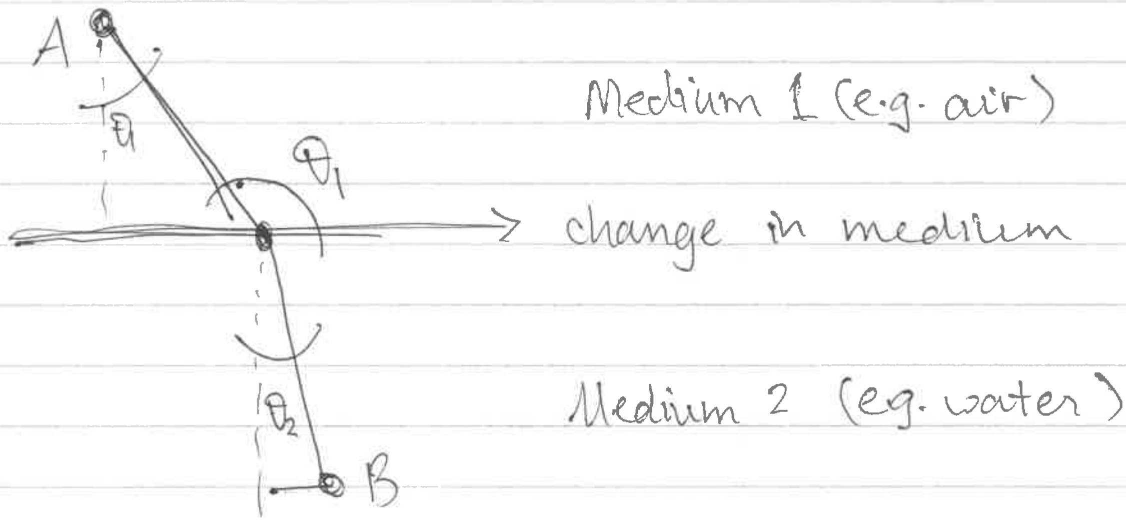
$A'(x)=0 \Rightarrow \frac{1}{2}(4-x^2)^{\frac{1}{2}} \cdot (*)$ — because this will simply things

$\Rightarrow (4-x^2) - x^2 \cdot (1) = 0$

$\rightarrow 2x^2-4=0 \Rightarrow x^2-2=0 \Rightarrow x=\sqrt{2}$

This critical point must be the abs max because the boundaries are ~~negati~~ zero and $A(x)$ is continuous.

EXAMPLE from physics $\longrightarrow$ is light.



"LAW": Angle θ_2 is such that travel time from A to B is minimized.

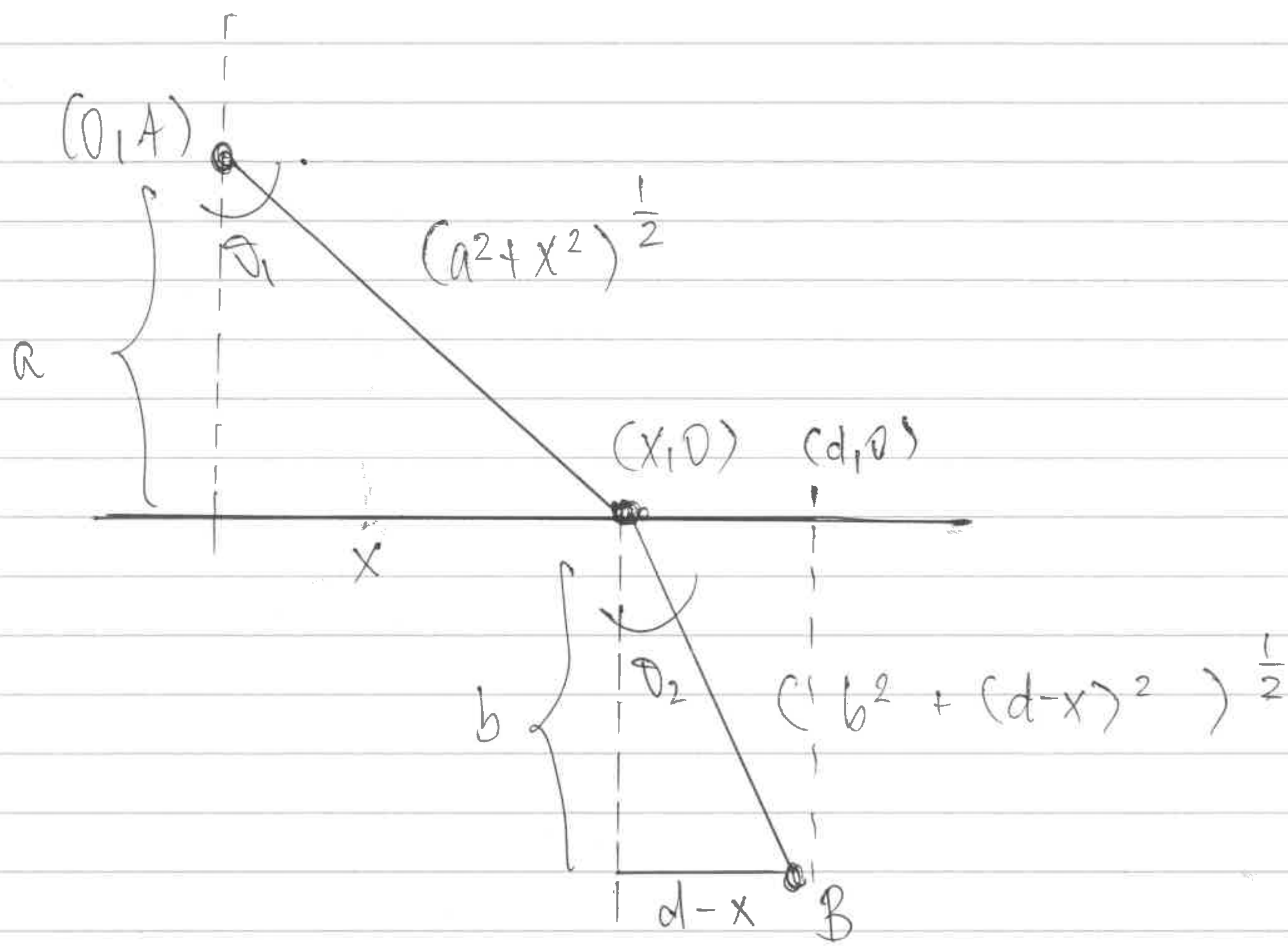
Use math to find θ_2 given θ_1.

Treat A and B as constant/given

Let c_1 = speed of light through medium 1

c_2 = " " medium 2.

c.



Time from A to B : (Note: rate · time = dist $\Rightarrow$ dist/rate = time

$$= T = \frac{(a^2+x^2)^{\frac{1}{2}}}{c_1} + \frac{(b^2+(d-x)^2)^{\frac{1}{2}}}{c_2}$$

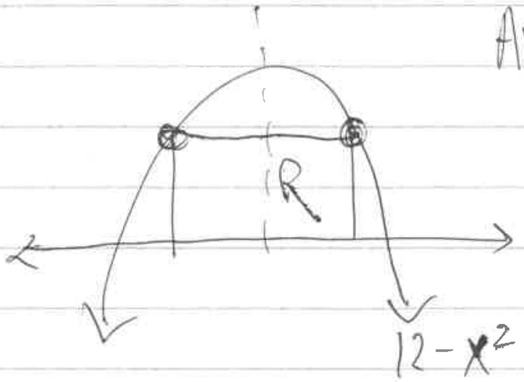
$$\Rightarrow T' = \frac{1}{2c_1}(a^2+x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2c_1}(b^2+(d-x)^2)^{-\frac{1}{2}} \cdot 2 \cdot (d-x) \cdot (-1)$$

$$= \frac{x}{c_1(a^2+x^2)^{\frac{1}{2}}} - \frac{(d-x)}{c_1(b^2+(d-x)^2)^{\frac{1}{2}}}$$

Snell's law.

$$= \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2} \Rightarrow \boxed{\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}}$$

EXAMPLE What is the largest area of the drawn rectangle R.



$\text{Area}(R) = A(x) = 2x(12-x^2) = 24x - 2x^3$

$x \in [0, \sqrt{12}]$

$\Rightarrow A' = 24 - 6x^2 = 6(6-x^2)$

$\Rightarrow A'' = -12x$

$\Rightarrow A'' < 0$ for our domain

$\rightarrow A$ is always CD

$A' = 0 \Rightarrow 0 = 6(6-x^2) \Rightarrow x^2 = 6 \Rightarrow x = \sqrt{6}$

Single crit pt is local max.

Endpoints: $A(0) = 0$, $A(\sqrt{12}) = 0$.

Therefore critical point is the absolute max.

Largest area: $A = 2\sqrt{6}\,(12-6) = 12\sqrt{6}$.

EXAMPLE: Find the point on the line

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (*)$$

that is closest to the origin.

~~D(x,y) = distance to origin.~~

$(*) \Rightarrow y = -\frac{b}{a}x - b$

Dist to origin = D.

$$D^2 = x^2 + y^2$$

$$\Rightarrow D^2 = x^2 + \left(-\frac{b}{a}x - b\right)^2 = x^2 + \left(\frac{b}{a}x + b\right)^2$$

$$\Rightarrow D = \left[x^2 + \left(\frac{b}{a}x + b\right)^2\right]^{\frac{1}{2}}$$

$$D' = \frac{1}{2} \frac{2x + 2\left(\frac{b}{a}x + b\right)\cdot\frac{b}{a}}{\left[x^2 + \left(\frac{b}{a}x+b\right)^2\right]^{\frac{1}{2}}} = \frac{x + \frac{b^2}{a^2}x + \frac{b^2}{a}}{\left[x^2 + \left(\frac{b}{a}x+b\right)^2\right]^{\frac{1}{2}}}$$

$$D' = 0 \Rightarrow 0 = x(1 + b^2/a^2) + b^2/a$$

$$\Rightarrow x = \frac{-b^2/a}{1 + b^2/a^2} = -\frac{b^2 a}{a^2 + b^2}$$

Note critical point cannot be global max because $D \to \infty$.

pt. closest to origin is:

$$\left(\frac{-b^2 a}{a^2+b^2},\ \frac{-b^3 a}{a(a^2+b^2)} - b \right)$$