§L'HOPITAL'S RULE

A strategy for finding limits of indeterminate form.

Defn Indeterminate Form

An indeterminate form is a meaningless expression which occassionally occur during limit calculations.

EXAMPLE: The following are indeterminate forms:
$0/0$, ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0, 1^∞
They are simply notations for a non-number.

Thm L'Hopitals Rule

When $f(a)/g(a) = 0/0$ OR ∞/∞. then

$$\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x)$$

assuming the RHS exists.

EXAMPLE

$$\lim_{x \to 0} \frac{3x - \sin x}{x} \quad w/ \quad \frac{3 \cdot 0 - \sin 0}{0} = \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}}{2x} = \frac{(\frac{1}{2})(-\frac{1}{2})(1+0)^{-3/2}}{2} = -1/8.$$

notation indicating LR was applied

EXAMPLE $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ w/ $\frac{0 - \sin 0}{0} = 0/0$

$$\overset{H}{=} \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \quad \text{w/} \quad \frac{1 - \cos 0}{3 \cdot 0} = \frac{1-1}{0} = 0/0$$

$$\overset{H}{=} \lim_{x \to 0} \frac{0 + \sin x}{6x} \quad \text{w/} \quad \frac{0 + \sin 0}{6 \cdot 0} = \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to 0} \frac{\cos x}{6} = 1/6.$$

EXAMPLE

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \overset{H}{=} \lim_{x \to 0} \frac{0 + \sin x}{1 + 2x} = \frac{0}{1} = 0.$$

which is not an indet. form

§ Indet Forms Involving ∞.

It will sometimes be necessary to manipulate the limit to obtain it.

EXAMPLE

$$\lim_{x\to\infty} x\cdot\sin\frac{1}{x}$$

$\infty\cdot\sin 0 = 0\cdot\infty$ indet — cannot yet apply LR.

let $x=\frac{1}{h}$ and note $x\to\infty$ when $h\to 0^+$.

$$=\lim_{h\to 0^+}\frac{\sin h}{h} \overset{H}{=} \lim_{h\to 0^+}\frac{\cos h}{1} = \frac{1}{1} = 1.$$

EXAMPLE $\lim\limits_{x\to 0^+}\sqrt{x}\ln x$ $\qquad 0\cdot -\infty$

$$=\lim_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}}$$

NOTE: $a\cdot b = b/\frac{1}{a}$

$$\overset{H}{=}\lim_{x\to 0^+}\frac{(\ln x)'}{(x^{-\frac{1}{2}})'} = \lim_{x\to 0^+}\frac{(1/x)}{(-\frac{1}{2}x^{-3/2})}$$

$$=\lim_{x\to 0^+}\frac{-2x^{3/2}}{x} = \lim_{x\to 0^+}-2x^{1/2} = 0.$$

EXAMPLE:

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) \qquad \frac{1}{0}-\frac{1}{0}=\infty-\infty$$

... cant use LR.

$$= \lim_{x\to 0}\frac{x-\sin x}{x\sin x} \overset{H}{=} \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} \qquad 0/0$$

$$\overset{H}{=} \lim_{x\to 0}\frac{0+\sin x}{\cos x+\cos x - x\sin x} = \frac{0+0}{1+1-0} = 0.$$

§ Indeterminate Powers

Notice:

$$\lim_{x\to a} \ln f(x) = L$$

$$\Rightarrow \lim_{x\to a} e^{\ln f(x)} = e^L$$

$$\Rightarrow \lim_{x\to a} f(x) = e^L$$

Rule

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e$$

Proof: Let $L = \lim_{x \to 0^+} (1+x)^{\frac{1}{x}}$ (*)

(*) $\Rightarrow \ln L = \lim_{x \to 0^+} \frac{1}{x} \cdot \ln(1+x) = \lim_{x \to 0^+} \frac{\ln(1+x)}{x}$: $\frac{0}{0}$

$= \lim_{x \to 0^+} \frac{\left(\frac{1}{1+x}\right)}{1} = 1 \Rightarrow \ln L = 1 \Rightarrow L = e$

$\Rightarrow \lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = e.$

Rule:

$$\lim_{x \to \infty} x^{\frac{1}{x}} = 1.$$

Proof: let $L = \lim_{x \to \infty} x^{\frac{1}{x}}$

$\Rightarrow \ln L = \lim_{x \to \infty} \frac{1}{x} \ln x = \lim_{x \to \infty} \frac{\ln x}{x}$: $\frac{\infty}{\infty}$ no LR yet...

let $x = \frac{1}{h}$ and note $x \to \infty$ when $h \to 0^+$

$= \lim_{h \to 0^+} \frac{\ln(\frac{1}{h})}{1/h}$: ∞/∞ $\overset{H}{=} \lim_{h \to 0^+} \frac{(\ln 1 - \ln h)'}{(1/h)'}$

$= \lim_{h \to 0^+} \frac{-1/h}{-1/h^2} = \lim_{h \to 0^+} h = 0.$ $\Rightarrow \ln L = 0 \Rightarrow L = 1.$

EXERCISES

A. $\lim_{x \to -2} \frac{x+2}{x^2-4}$

B. $\lim_{x \to \infty} \frac{1-\cos x}{x^2}$

C. $\lim_{t \to -3} \frac{t^3-4t+15}{t^2-t-12}$

D. $\lim_{x \to 0^+} x \tan\left(\frac{\pi}{2}-x\right)$

E. $\lim_{\theta \to 0} \frac{3^{\sin\theta}-1}{\theta}$

F. $\lim_{x \to 0^+} \frac{x}{e^{-1/x}}$

EXERCISE Show $\lim_{K \to \infty} \left(1+\frac{r}{K}\right)^K = e^r$.