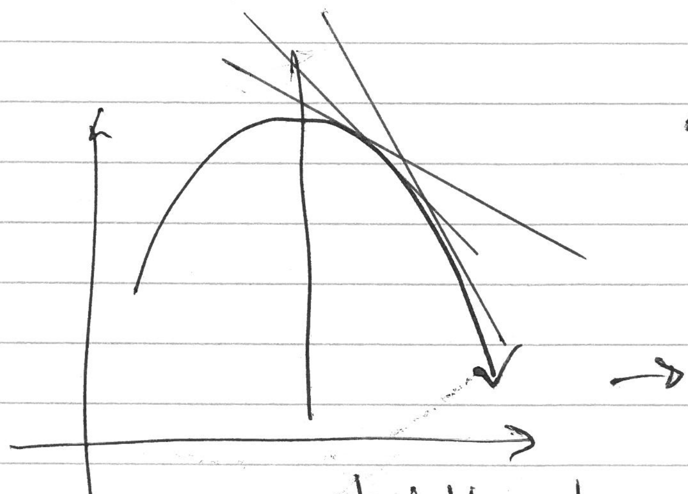
§ Monotonic Functions

English: Functions that are everywhere strictly increasing / decreasing in their domain.

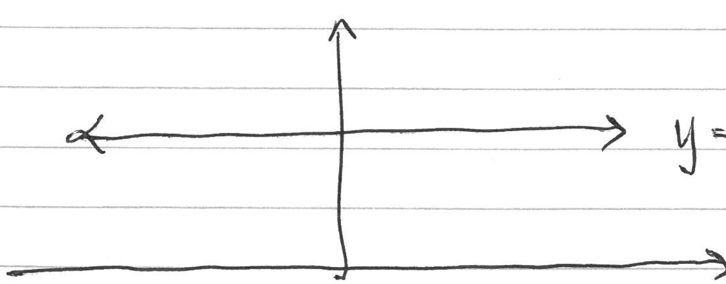
Geometry

tangent is everywhere pos



tangent is everywhere negative.

strictly decreasing



Neither increasing nor decreasing.

Immediate Consequence of

<u>Corollary</u> Provided

- f is cont on $[a,b]$
- f is diffable on (a,b)

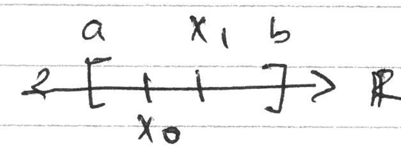
f is <u>increasing</u> on $[a,b] \Longleftrightarrow \forall x \in (a,b);\ f'(x) > 0$

f is <u>decreasing</u> on $[a,b] \Longleftrightarrow \forall x \in (a,b);\ f'(x) < 0$

<u>Proof</u> (increasing)

Suppose $\forall x \in (a,b);\ f'(x) > 0$.

Take $x_0, x_1 \in [a,b]$ w/ $x_0 < x_1$. :



MVT $\Rightarrow \exists c \in (x_0, x_1) : f'(c) = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$\Rightarrow \quad f'(c)(x_1 - x_0) = f(x_1) - f(x_0)$

But... $f'(c) > 0$ by assumption. Also $x_0 < x_1 \Rightarrow x_1 - x_0 > 0$

$\Rightarrow \quad f'(c)(x - x_0) > 0 \quad \Rightarrow \quad f(x_1) - f(x_0) > 0$

$\Rightarrow \quad f(x_1) > f(x_0)$.

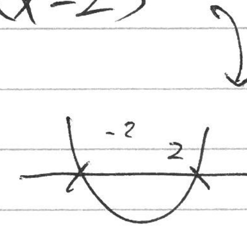
Thus $x_0 < x_1 \Rightarrow f(x_0) < f(x_1)$

which means f is <u>increasing</u> by defn

<u>EXAMPLE</u> Where is $x^3 - 12x - 5 = y$ incr? and decr?

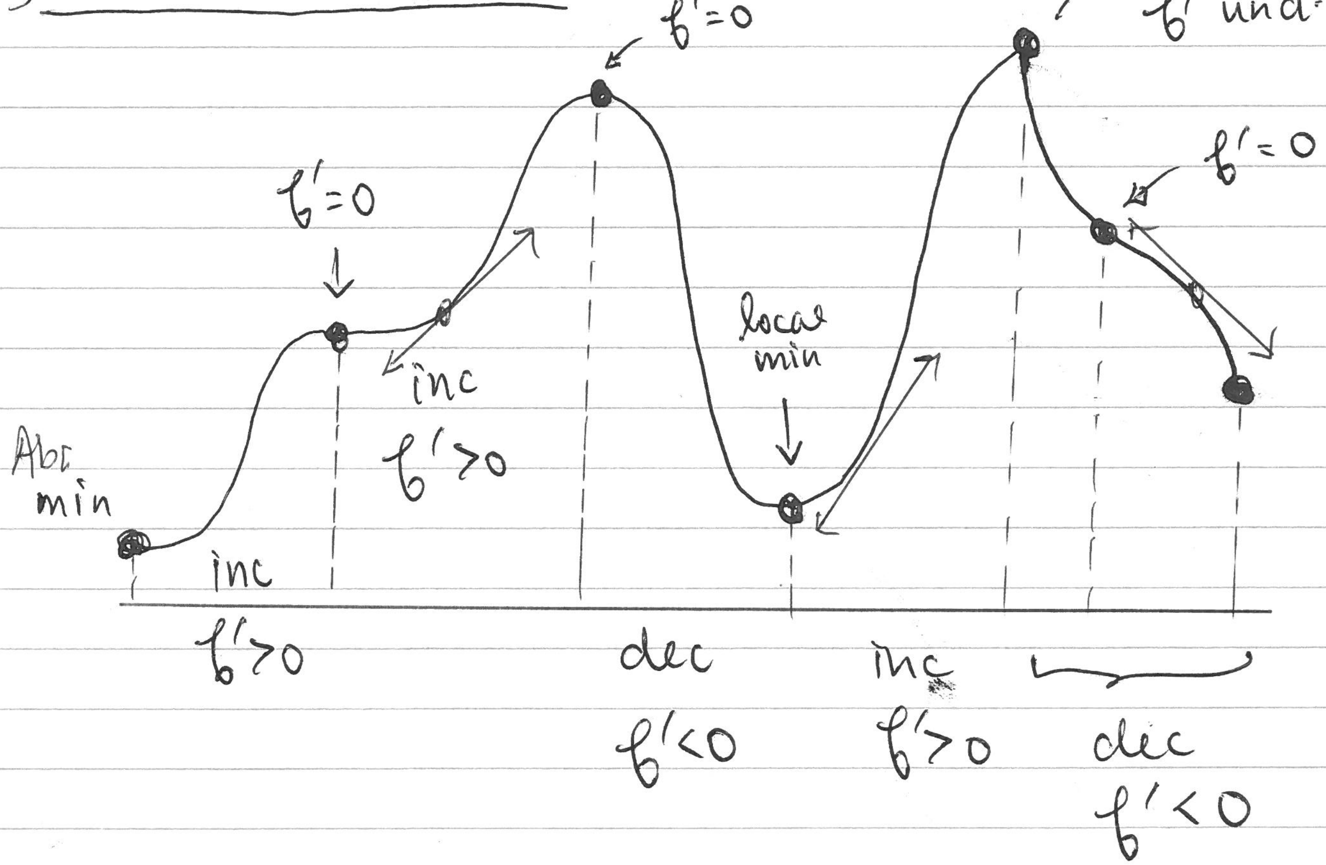
$y' = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$

$f'(x)$: +ve on the left of -2, −ve between -2 and 2, +ve on the right of 2

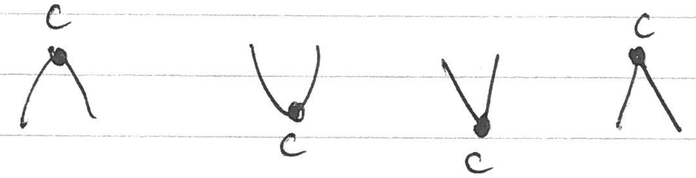


f inc on $[-\infty, -2]$
dec on $[-2, 2]$
inc on $[2, \infty)$

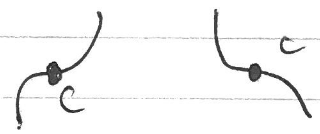
§ First Derivative Test



Observation: local extrema occur at



But <u>not</u>



Though in all cases $f'(c) = 0$ or UND

<u>FIRST DERIVATIVE TEST</u>

Provided
- c is a critical point of f
- f is continuous on $[a,b]$
- f is diffable on $(a,b) - \{c\}$ then...

CASE $\vee$ $f'(c-\varepsilon) < 0$ and $f'(c+\varepsilon) > 0$ for $\varepsilon > 0$ tiny

$\Rightarrow (c, f(c))$ is a local minima.

$\wedge$ $f'(c-\varepsilon) > 0$ and $f'(c+\varepsilon) < 0$ for $\varepsilon > 0$ tiny

$\Rightarrow (c, f(c))$ is a local <u>maxima</u>.

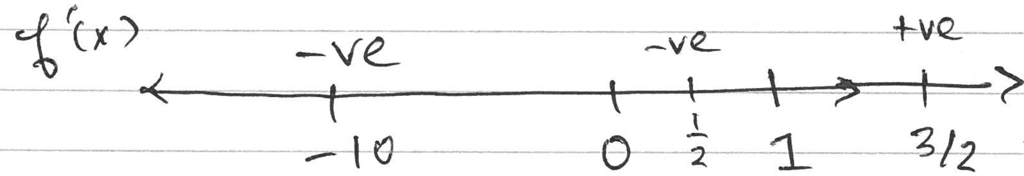
Otherwise: $(c, f(c))$ is <u>not</u> a local extrema.

EXAMPLE Find all local extrema of

$$f(x) = x^{\frac{1}{3}}(x-4) = x^{4/3} - 4x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}} = \frac{4}{3}x^{-2/3}(x-1)$$

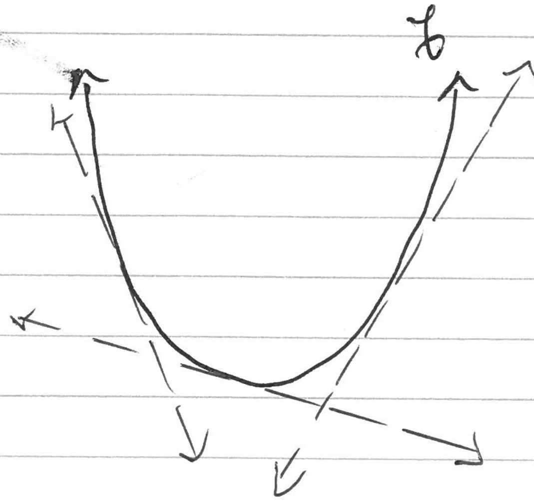
So $f'(1) = 0$ and $f'(0) = \text{UND} \Rightarrow (0, f(0))$, $(1, f(1))$ are critical points.



~~Concavity and Curve Sketching~~

§Concavity and Curve Sketching

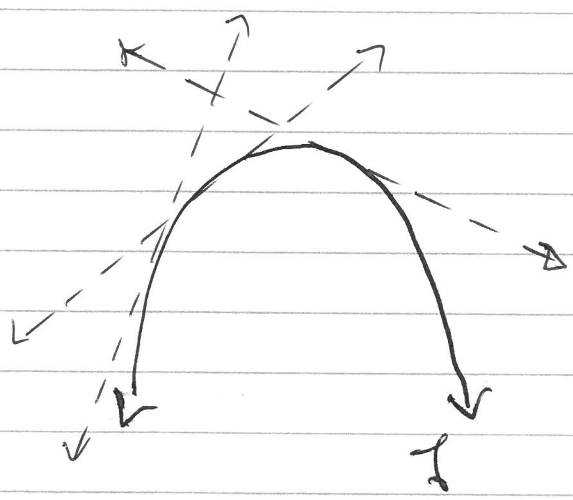
Concave Up



Tangent slope everywhere inc.
$\Rightarrow f'(x)$ is incr.

$\Rightarrow f''(x) > 0$

Concave Down



Tangent slope everywhere decreasing
$\Rightarrow f'(x)$ is decr.

$\Rightarrow f''(x) < 0$

Def$^{\underline{n}}$ Concave Up on $I = (a,b)$

f is diffable and f' increasing on I.

Def$^{\underline{n}}$ Concave Down on $I = (a,b)$

f is diffable and f' decreasing on I.

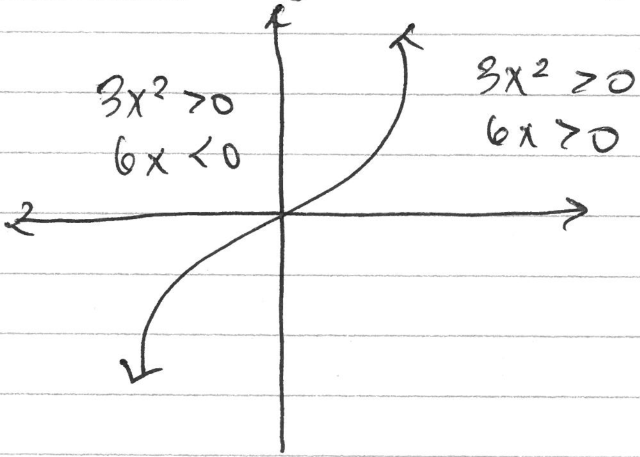
Second Derivative Test

For $I = (a,b) \subseteq \mathbb{R}$

$$f'' > 0 \text{ on } I \Rightarrow f \text{ is CU on } I$$

$$f'' < 0 \text{ on } I \rightarrow f \text{ is CD on } I$$

EXAMPLE $y = x^3 \rightarrow y' = 3x^2 \Rightarrow y'' = 6x$



$(0,0)$ is critical point
$[0, \infty]$ is CU
$(-\infty, 0)$ is CD

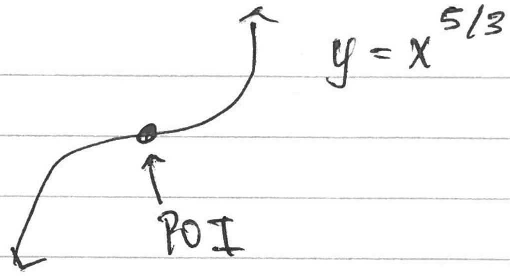
Def^n Point of Inflection POI

A point $(c, f(c))$ where concavity changes and $f'(c)$ exists is called a <u>point of inflection</u>.

Prop^n $(c, f(c))$ is a POI

$$\Leftrightarrow f''(c) = 0 \text{ or } f''(c) \text{ DNE}$$

EXAMPLE



$y' = 5/3\, x^{2/3} = 0$ when $x = 0$

$y'' = \frac{2\cdot 5}{3\cdot 3} x^{-1/3}$ is und at $x = 0$.

Thus $(0,0)$ is a POI.

SECOND DERIVATIVE TEST

Provided • f'' is cont on (a,b) and $c \in (a,b)$

Then

(1) $f'(c) = 0$ and $f''(c) < 0$
$\Rightarrow$ $(c, f(x))$ is local maxima.

(2) $f'(c) = 0$ and $f''(c) > 0$
$\Rightarrow$ $(c, f(c))$ is local minima

(3) $f'(c) = 0$ and $f''(c) = 0$
$\Rightarrow$ inconclusive

§ Oblique/Slant Asymptotes

When a rational function $f(x)/g(x)$ satisfies $f(x), g(x)$ are polynomials w/ $\deg f - \deg g = 1$ it will have a slant asymptote which is useful for sketching

~~EXAM~~ Strategy for finding slant asymptotes:

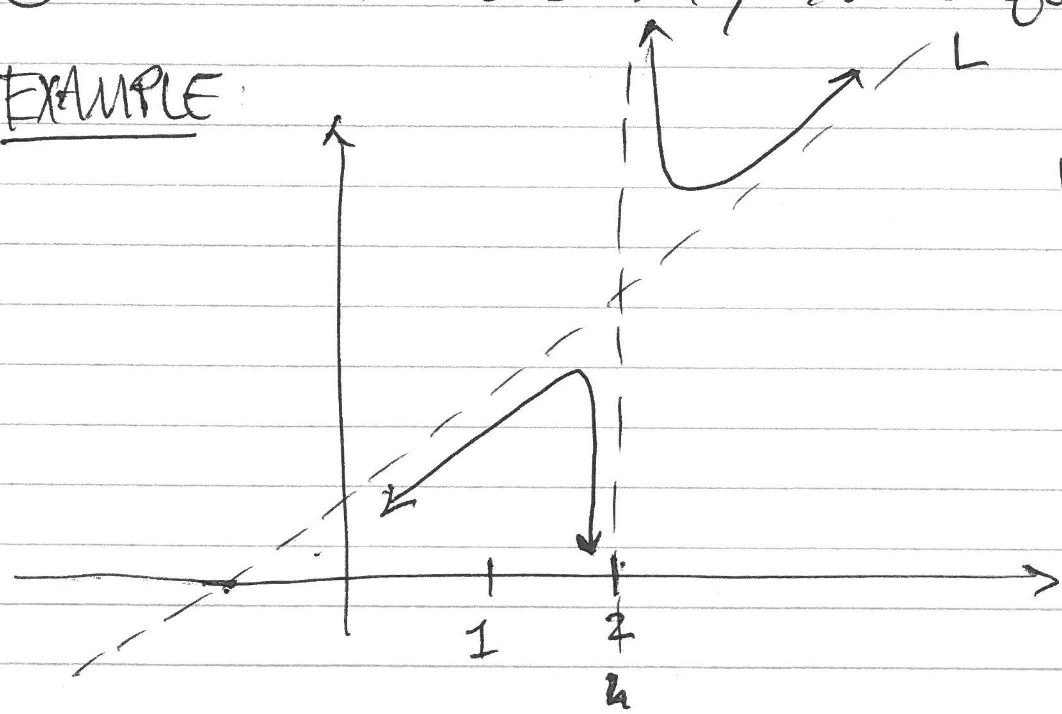
Provided $\deg f - \deg g = 1$ ~~let~~

① ~~let~~ $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} mx + b$

② solve for m.

③ backsubstitute m, solve for b

EXAMPLE:



$y = \frac{x^2 - 3}{2x - 4}$

Find L.

$$\lim_{x \to \infty} \frac{x^2-3}{2x-4} = \lim_{x \to \infty} mx + b$$

$$\Rightarrow \lim_{x \to \infty} x^2/2x = \lim_{x \to \infty} mx$$

$$\Rightarrow \quad m = \lim_{x \to \infty} x^2/2x^2 = 1/2.$$

Sub $m = 1/2$

$$\lim_{x \to \infty} \frac{x^2-3}{2x-4} = \lim_{x \to \infty} \frac{1}{2}x + b$$

$$\Rightarrow b = \lim_{x \to \infty} \frac{x^2-3}{2x-4} - \frac{x}{2}$$

$$= \lim_{x \to \infty} \frac{x^2-3-x(x-2)}{2x-4}$$

$$= \lim_{x \to \infty} \frac{x^2-3-x^2+2x}{2x-4}$$

$$= \lim_{x \to \infty} \frac{2x-3}{2x-4} = 1$$

$$\boxed{L: \ y = \frac{1}{2}x + 1.}$$

EXERCISE: Find HA/VA/OA

Confirm w/ DESMOS

- $y = \dfrac{x^2}{x-1}$
- $y = \dfrac{x^2+1}{x-1}$
- $y = \dfrac{x^2-1}{2x+4}$
- $y = \dfrac{x^3+1}{x^2}$

§ Curve Sketching

Use as much algebra to "guess" the shape of the graph.

Find:
- Intervals of inc/dec.
- Local min/max
- Concavity (i.e. curvature)
- Asymptotes (HA, VA, OA)
- Roots
- Critical Points
- Inflection points
- Intercepts (y-int, x-int)

<u>EXAMPLE</u> Plot $f(x) = \dfrac{x^2+4}{2x}$

$x \notin$ dom f $\Rightarrow$ asymptote

<u>X-int</u> (Roots)

$x^2+4 \neq 0 \Rightarrow$ no roots
i.e. should never cross x-axis

<u>y-int</u> None. $x \notin$ dom f.

<u>Asymptotes</u>

VA: $\lim\limits_{x \to 0^+} f(x) = +\infty$, $\lim\limits_{x \to 0^-} f(x) = -\infty$

HA: $\lim\limits_{x \to \infty} f(x) = \infty$, $\lim\limits_{x \to -\infty} f(x) = -\infty$

Slant asymptote:

$\lim\limits_{x \to \infty} \dfrac{x^2+4}{2x} = \lim\limits_{x \to \infty} mx+b$

$\Rightarrow \lim\limits_{x \to \infty} x^2/2x = \lim\limits_{x \to \infty} mx \Rightarrow m = \lim\limits_{x \to \infty} x^2/2x^2 = \frac{1}{2}$.

Recover b

$$\lim_{x \to \infty} \frac{x^2+4}{2x} = \lim_{x \to \infty} \frac{1}{2}x + b$$

$$\Rightarrow b = \lim_{x \to \infty} \frac{x^2+4}{2x} - \frac{x}{2}$$

$$= \lim_{x \to \infty} \frac{x^2+4-x^2}{2x} = \lim_{x \to \infty} \frac{2}{x} = 0$$

OA: $y = \frac{1}{2}x$

Critical points

$$f'(x) = \frac{2x(2x) - (x^2+4)2}{4x^2} = \frac{2x^2 - x^2 - 4}{2x^2} = \frac{x^2-4}{2x^2}$$

$$= \frac{(x+2)(x-2)}{2x^2}$$

$f'(x) = 0 \quad x = 2, -2$ crit pts

$f'(x) = \text{UND} \quad x = 0$ $= \{(2,2), (-2,-2)\}$

f' +ve | −ve | −ve | +ve

inc -2 dec 0 inc 2 dec

$$f'' = \frac{2x(2x^2) - (x^2-4)(4x)}{4x^4}$$

$$= \frac{4x^3 - 4x^3 + 16x}{4x^4} = \frac{4}{x^3}$$

f'' = UND at $x=0$

f'' −ve | +ve

CD 0 CU

Combining Everything