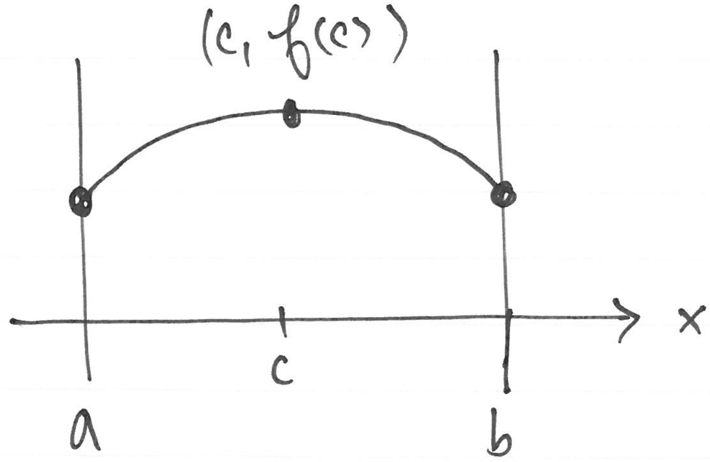
§ EXTREME VALUES OF FUNCTIONS

Geometry:



This point is called the abs-maximum of f in $[a,b]$. Intuitively it is the biggest f can be in the interval.

Def$^{\underline{n}}$ Absolute Maximum

f is a function and $D \subseteq \text{dom} f$ is an interval. ~~$c \in D$ is~~ We say "f has an absolute maximum at $c \in D$" when

$$\forall x \in D;\ f(x) \leq f(c).$$

Def$^{\underline{n}}$ Absolute Minimum

...

$$\forall x \in D;\ f(x) \geq f(c)$$

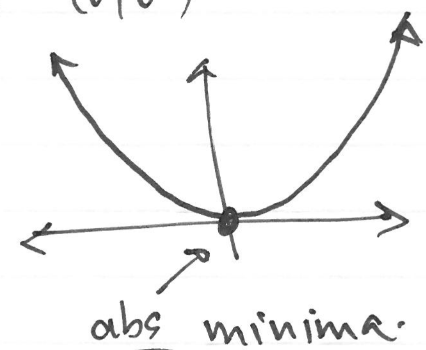
EXAMPLE: What is the ~~[illegible]~~ abs min/max of $f(x) = 5$?

<u>Answer</u> Abs max $= 5$. Abs min $= 5$. — i.e. a value <u>can</u> be <u>both</u> max & min.

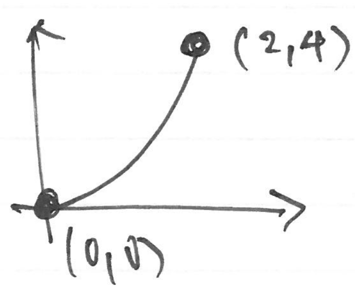
QUESTION: Does abs max/min always exist?

<u>Answer</u> No! x^2 has no maxima on $(-\infty, \infty)$.
It <u>does</u> have minima at $(0,0)$.

Continuous functions ~~Functions~~ over <u>closed</u> intervals <u>for sure</u>.



<u>EXAMPLE</u> $x^2 = f(x)$ on $[0, 2]$



— abs max

— abs min

Thm Extreme Value Thm (EVT)

All continuous functions f over $[a,b]$ obtain **both** absolute minima/maxima.

Namely $\exists x_0, x_1 \in [a,b]$:

$(x_0, f(x_0))$ is an abs minima,

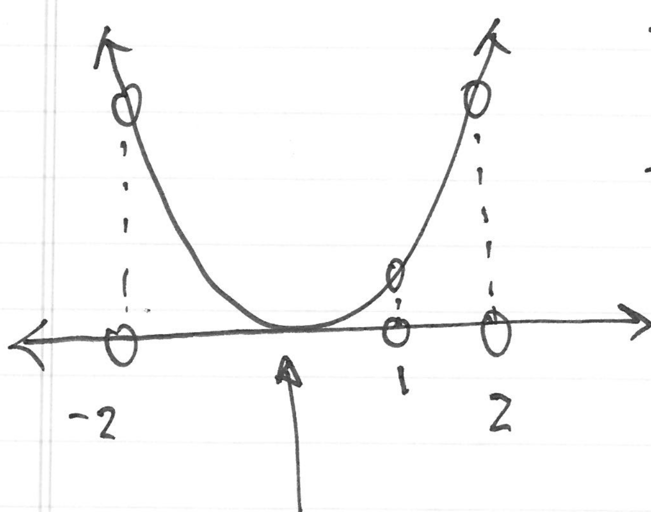
$(x_1, f(x_1))$ is an abs maxima

Proof Omitted.

WARNING: Interval must be closed to use EVT

QUESTION: When does a function have a max/min on an open interval?

- No min/max on $(1,2)$
- Minima ~~of~~ at $0 \in (-2,2)$



abs minima on $(-2,2)$.

Extrema on (small) open intervals are called <u>local</u> maxima/minima.

<u>Def</u>

<u>Def$^{\underline{n}}$</u> Local Maxima

A point ~~$c \in \text{dom} f$~~ $(c, f(c)) \in G(f)$ is called a <u>local maxima</u> when $\exists a, b \in \mathbb{R}$: $(a,b) \subseteq \text{dom} f$ and $c \in (a,b)$: $f(x) \leq f(c) \ \forall x \in (a,b)$.

<u>Def$^{\underline{n}}$</u> Local Minima

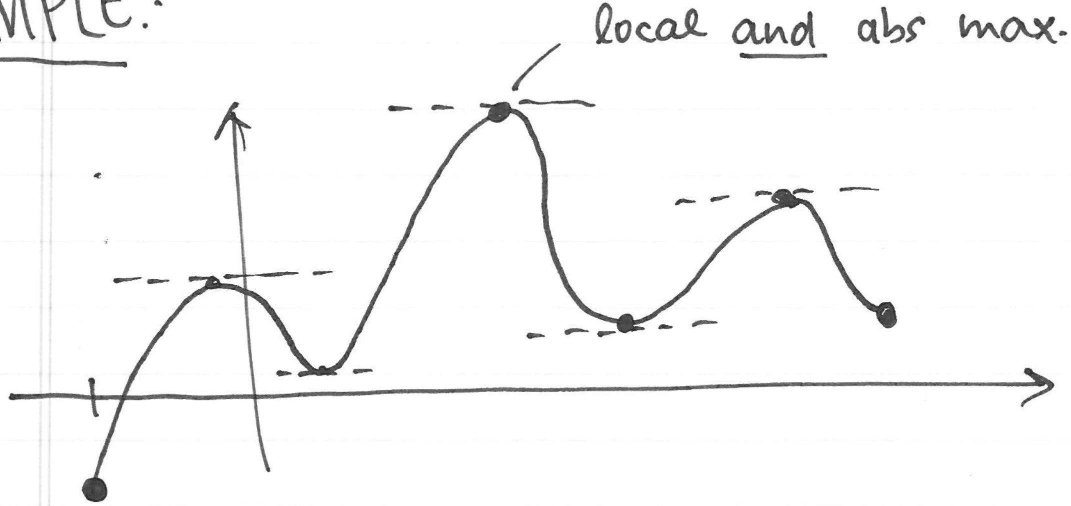
... $f(x) \geq f(c) \ \forall x \in (a,b)$.

<u>Def$^{\underline{n}}$</u> Interior Point

An <u>interior point</u> of an interval $[a,b] \subseteq \mathbb{R}$ is any point $c \in (a,b)$.

Effectively this means c cannot be on the boundary.

EXAMPLE:



Notice Tangets are flat at local extrema.

§ Finding Extrema

Thm If $(c, f(c))$ is a local extrema on f then $f'(c) = 0$.

Proof Consider the slope of the tangent at $c \in \operatorname{dom} f$.

$$\cancel{f'(c) = \lim_{h \to 0} \frac{f(c) - f(c+h)}{\quad}}$$

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Assume, without loss of generality, that there is a local minimum $\Rightarrow$

$$f(c+h) \geq f(c)$$
$$\Rightarrow f(c+h) - f(c) \geq 0.$$

Thus $\lim\limits_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h}$ — always positive / — always positive

$\lim\limits_{h \to 0^-} \dfrac{f(c+h) - f(c)}{h}$ — always positive / — always negative

$\Rightarrow \quad f'(c) \leq 0$ and $f'(c) \geq 0$

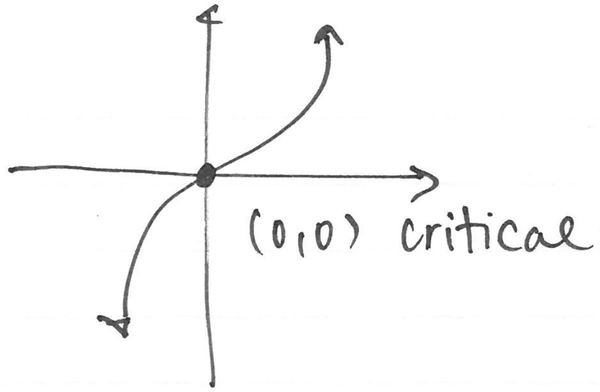
$\Rightarrow \quad f'(c) = 0.$

(We can repeat this argument for c a local maxima.)

Def$^{\underline{n}}$ Critical Point

A point $c \in \operatorname{dom} f$ where $f'(c) = 0$ or UNDEFINED is called a critical point.

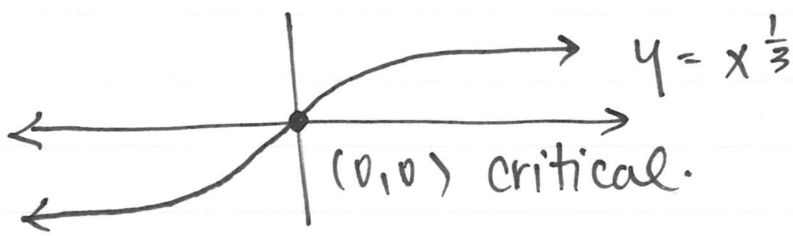
EXAMPLE: $y = x^3$



$\Rightarrow y' = 3x^2 = 0$

$\Rightarrow x = 0$ is a critical point.

EXAMPLE: $y = x^{\frac{1}{3}} \Rightarrow y' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}} \Rightarrow x = 0$ is a crit pt.



EXAMPLE: Find all max/min on

$$f(x) = 10x(2 - \ln x) \text{ on } [1, e^2]$$

All extrema are at critical points and <u>maybe</u> end-points

$f'(x) = 10(2 - \ln x) + 10x\left(-\frac{1}{x}\right) = 0$

$\Rightarrow 0 = 2 - \ln x - 1 \Rightarrow \ln x = 1 \Rightarrow x = e.$

CRITICAL POINTS: $(e, f(e)) = (e, 10e)$

ENDPOINTS: $(1, f(1)) = (1, 20)$

$(e^2, f(e^2)) = (e^2, 10e^2(2-2)) = (e^2, 0)$

Note: $e > 2 \Rightarrow 10e > 20$

$(e, \cancel{f(e)}^{10e})$ — absolute maximum

$(1, \cancel{f(1)}^{20})$ — nothing special

$(e^2, 0)$ — absolute minimum.

... in $[1, e^2]$

EXERCISE: Find abs max/min on

- $y = x^{2/3}$ $\quad x \in [-2, 3]$
- $f(x) = |x|$ $\quad x \in [-1, 1]$
- $g(x) = \begin{cases} -x & x \in [0,1) \\ x-1 & x \in [1,2] \end{cases}$

Verify your answers w/ DESMOS.

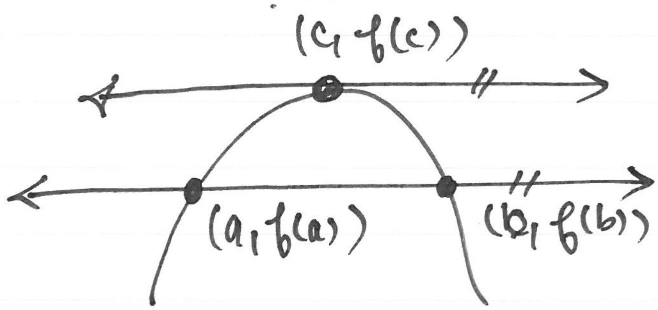
EXERCISE Find critical points and abs/min/max.

for

- $y = x^{2/3}(x+2)$
- $y = x^2\sqrt{3-x}$

§ MEAN VALUE THM

NOTICE :



move down a tad.

We find a secant of slope zero (parallel to the tangent). Thus if there is a, b : $f(a) = f(b)$ we know there must be some $c \in (a,b)$ where the tangent is horizontal.

Thm Rolle's.

Provided
- $f(x)$ is continuous over $[a,b]$,
- $f(x)$ is diffable over (a,b).

$$f(a) = f(b) \Rightarrow \exists c \in (a,b) : f'(c) = 0.$$

Proof : Omitted.

Using Rolle's Thm...

EXAMPLE: Show

$$f(x) = x^3 + 3x + 1 = 0$$

has exactly one solution.

Notice • $f(0) = 1 > 1$ and $f(-1) = -3 < 0$

• f is a polynomial $\Rightarrow$ everywhere cont.

~~By IVT $\Rightarrow \exists c \in (0,$~~

By IVT $\Rightarrow \exists c \in (-1, 0) : f(c) = 0$.

However, this does not exclude the possibility of several c's. We must show c is the only solution.

Suppose, towards a contradiction, $\exists c_0, c_1,\ c_0 \neq c_1$:

$$f(c_0) = 0 = f(c_1)$$

Rolle's $\Rightarrow \exists a \in (c_0, c_1) : f'(a) = 0$

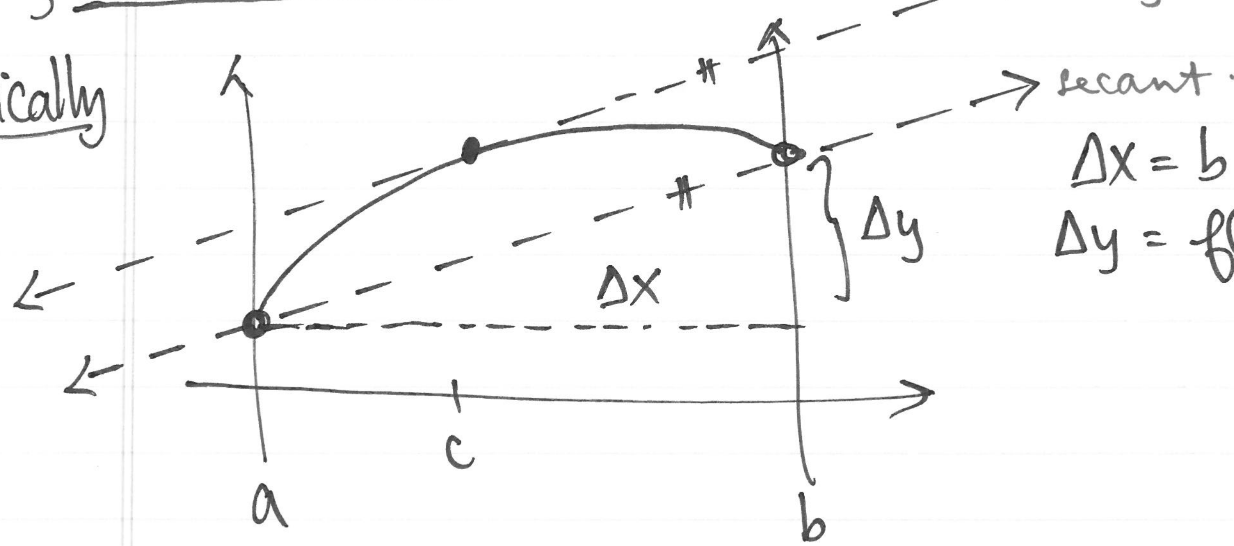
But $f'(x) = 3x^2 + 3$ so $f'(x) > 0$

$\rightarrow \exists a : 3a^2 + a = 0$ $\Rightarrow\Leftarrow$ (Impossible)

Thus solution is unique.

§ Mean Value Thm

Basically



$\Delta x = b - a$
$\Delta y = f(b) - f(a)$.

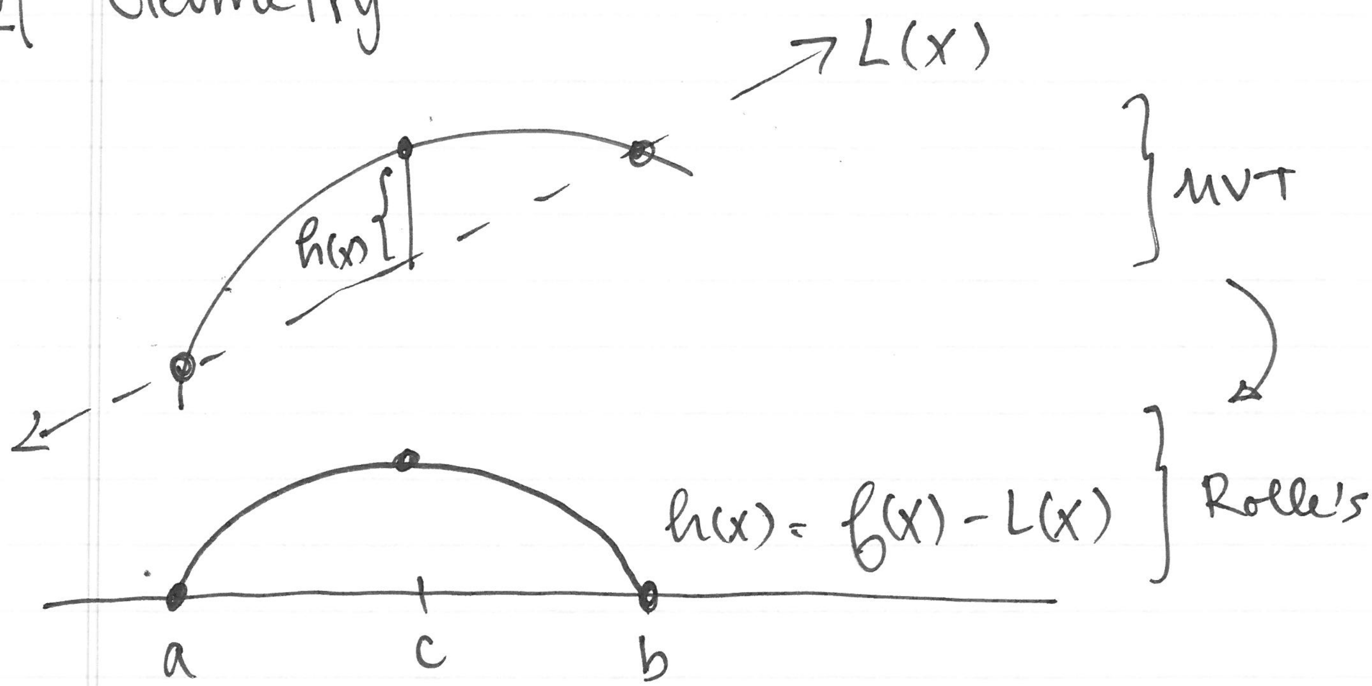
Thm MEAN VALUE (MVT)

Provided
- f is cont on $[a, b]$
- f is diffable on (a, b)

$$\exists c \in (a,b) : f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically: There is a tangent line at $(c, f(c))$ parallel to the secant connecting $(a, f(a))$ to $(b, f(b))$.

Proof Geometry

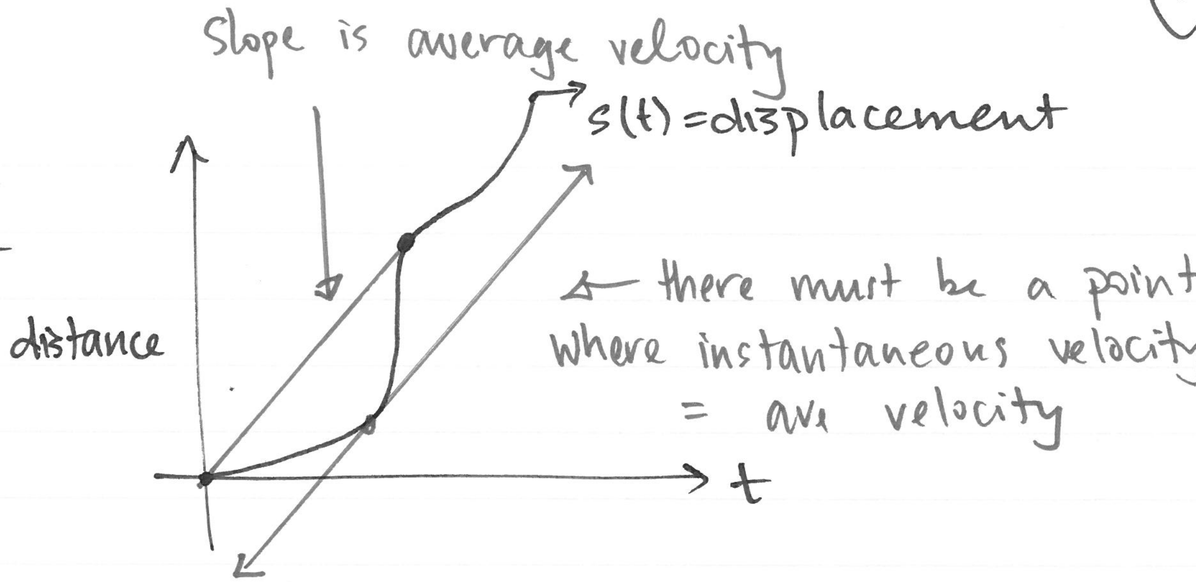


And now use Rolle's to get c and show it has the desired property.

EXAMPLE It took Alice $\frac{1}{2}$hr to drive 10Km. Her initial speed was 0Km/hr. Prove Alice had instantaneous velocity 20Km/hr at some point in her journey.

Can you do the same for 100Km/hr?

EXAMPLE



EXERCISE Suppose f'' is continuous on $[a,b]$. and f has three zeroes in $[a,b]$.

Show f'' has at least one zero in (a,b).

EXERCISE: For what $a, m, b \in \mathbb{R}$ does

$$f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & x \in (0,1) \\ mx + b & x \in [1,2] \end{cases}$$

satisfy conditions for MVT?