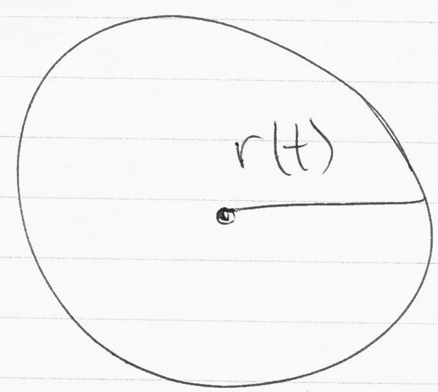
# § Related Rates

**MOTIVATION**: A circle's radius expands at 2m/s starting from 0m. How quickly is the area increasing at any given time?




$r(t)$ = radius at time t

$\Rightarrow \frac{dr}{dt}$ = how rapidly r is increasing at t. = 2m/s

$A(t) = \pi(r(t))^2$ = area at time t

$\rightarrow \frac{dA}{dt} = 2\pi\, r(t) \cdot \frac{dr}{dt}$ = how rapidly area is changing at any given time

$= 4\pi r(t)$ in this example.

Volume of sphere = $V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dt} = 4 \cdot \pi r^2 \cdot \frac{dr}{dt} = 2\frac{dA}{dt}$ $\Rightarrow$ Volume of a sphere increases at double the rate of the area of the circle inside it.

<u>EXAMPLE</u> $x = y^3 - y$, $\frac{dx}{dt} = 5 \Rightarrow \frac{dy}{dt} = ?$ at $y = 2$

$$x = y^3 - y \Rightarrow \frac{dx}{dt} = 3y^2 \cdot \frac{dy}{dt} - \frac{dy}{dt}$$

$$\Rightarrow 5 = (3 \cdot 2^2 - 1)\, dy/dt$$

$$\Rightarrow dy/dt = 5/11$$

<u>EXAMPLE</u> $L = \sqrt{x^2 + y^2}$, $\frac{dx}{dt} = -1$, $\frac{dy}{dt} = 3$

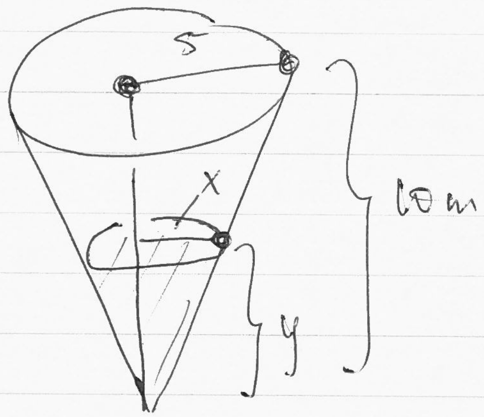
$\Rightarrow \frac{dL}{dt} = ?$ at $x = 5$, $y = 12$.

$$\frac{dL}{dt} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}\left(2x \cdot \frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

$$= \frac{1}{2(5^2 + 12^2)^{\frac{1}{2}}} \cdot (2 \cdot 5 \cdot (-1) + 2 \cdot 12 \cdot 3)$$

$$= 31/13$$

**EXAMPLE** Water fills the pictured cone at $9m^3/s$. How fast is the water level rising when the water is 6m deep?




Recall – volume of cone $= \frac{1}{3}\pi h^2 r$.

**HAVE** $\frac{dV}{dt} = 9m^3/s$ $\quad y = 6m$. **WANT** $\frac{dy}{dt}$

**RELATE** ~~$\frac{dV}{dt} = /$~~ $V = \frac{1}{3}\pi x^2 y$ = volume of liquid.

$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{3}\left(2x\frac{dx}{dt}y + x^2\boxed{\frac{dy}{dt}}\right) \quad (*)$$

Need $x, y, \frac{dV}{dt}, \frac{dx}{dt}$

By triangles $\frac{10}{5} = \frac{y}{x} \Rightarrow 2x = y \Rightarrow 2\frac{dx}{dt} = \frac{dy}{dt}$

$$\Downarrow$$

$$x = \frac{1}{2}\cdot 6 = 3$$

(*) becomes...

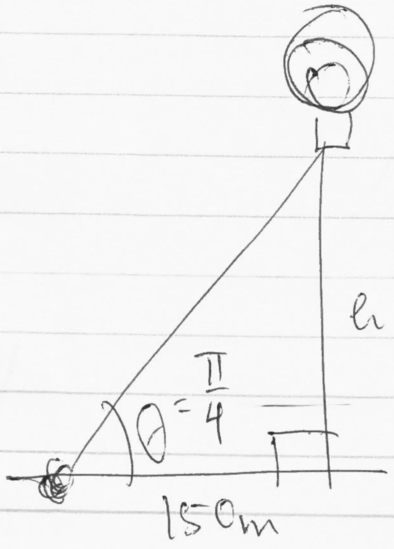
$$\frac{dV}{dt} = \frac{\pi}{3}\left(2 \cdot xy \frac{dy}{dt} + x^2 \frac{dy}{dt}\right)$$

$$\Rightarrow \quad 9 = \frac{\pi}{3}\left(3 \cdot 6 + 3^2\right)\frac{dy}{dt}$$

$$\Rightarrow \quad dy/dt = \frac{1}{\pi} \text{ m/s.} \quad @ \quad y = 6m.$$

<u>EXAMPLE</u> A hot air balloon rising from a field is tracked by a laser pen 150m from the launching position. At the moment the tracking angle is $\theta = \pi/4$ how fast is the balloon rising ~~at~~ given that the tracking angle is increasing at 0.14 rad/min.

<u>HAVE</u> $h = 150 \tan\theta$ $\quad \frac{d\theta}{dt} = \frac{14}{100} = \frac{7}{50}$ rad/min.




<u>WANT</u> $\frac{dh}{dt}$ @

<u>RELATE</u> (HAVES & WANTS)

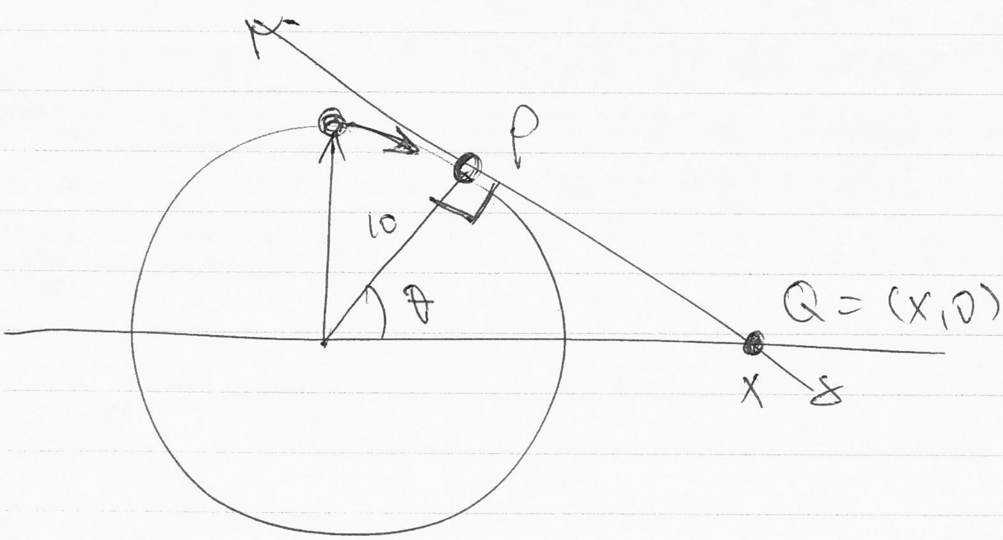
$\tan\theta = \frac{h}{150} \Rightarrow h = 150\tan\theta$

$\Rightarrow \frac{dh}{dt} = 150\sec^2\theta \frac{d\theta}{dt}$ at $\theta = \pi/4$, $h = 150\tan\frac{\pi}{4}$, $\frac{d\theta}{dt} = 7/50$ rad/min

NOTE $\sec\frac{\pi}{4} = \sqrt{2}$

$= 150 \cdot 2 \cdot 7/50 = 42$ !

EXAMPLE




It takes 30 seconds for P to move from (0, 10) to (10, 0) along the circle.

How fast is Q moving (in its x-component) towards the origin when it is 20m away from it.

HAVE: $\frac{d\theta}{dt} = \frac{\pi/2}{30} \frac{\text{rad}}{\text{sec}} = -\pi \frac{\text{rad}}{\text{min}}.$

~~$x = 20m$~~

WANT: $\frac{dx}{dt}$ @ $x = 20m$

RELATE: $\cos\theta = \frac{10}{x} \Rightarrow x\cos\theta = 10$

$\Rightarrow \boxed{\frac{dx}{dt}} \cos\theta - x\sin\theta \frac{d\theta}{dt} = 0$

$\Rightarrow \frac{dx}{dt} = x \frac{\sin\theta}{\cos\theta} \cdot \frac{d\theta}{dt}$

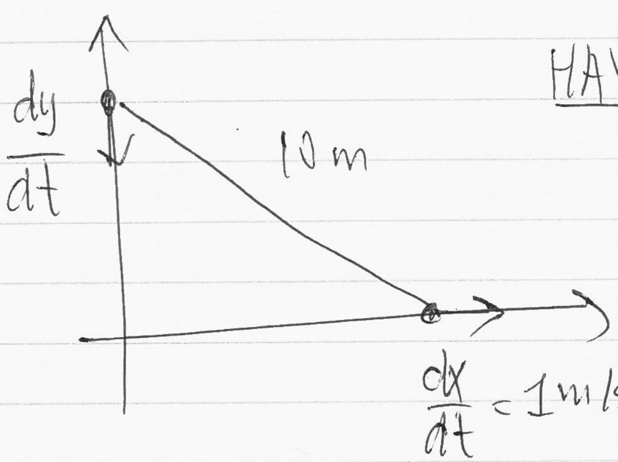
$\Rightarrow \frac{dx}{dt} = x\tan\theta \cdot \frac{d\theta}{dt} = 20m$

When $x=20$ $\tan\theta = \sqrt{20^2-10^2}/10 = \sqrt{300}/10 = \sqrt{3}$

$\Rightarrow \frac{dx}{dt} = 20 \cdot \sqrt{3} \cdot (-\pi) = -20\sqrt{3}\cdot\pi$ m/min.

$$\boxed{\frac{dx}{dt} = -20\sqrt{3}\cdot\pi}$$

EXAMPLE: A 10m ladder slides down a wall. If the bottom of the ladder slides away from the wall at 1m/s, how fast is the top of the ladder travelling towards the ground when it is 6m from the ground?




HAVE: $\frac{dx}{dt} = 1 m/s$ ladder = 10m

WANT: $\frac{dy}{dt}$ @ $y = 6m$

RELATE: $x^2 + y^2 = 10^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$

$\Rightarrow \frac{dy}{dt} = \frac{-8}{6} = \frac{-4}{3} m/s$

Speed = 4/3 m/s