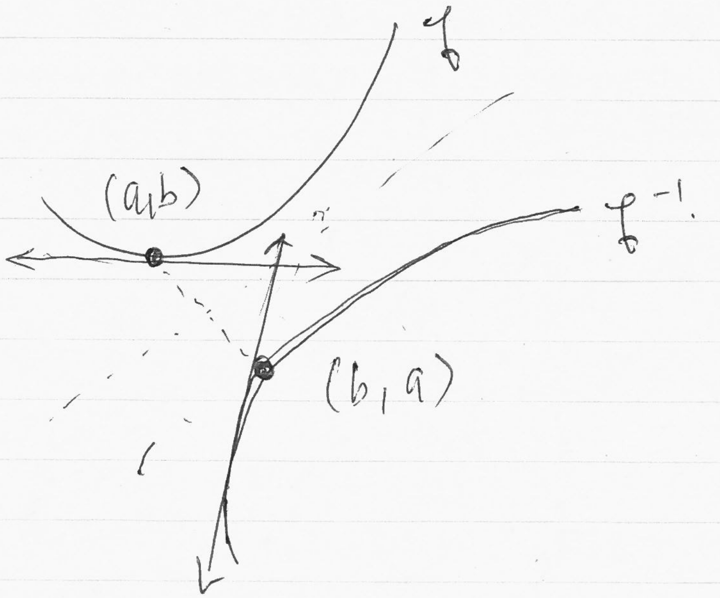
§ DERIVATIVES OF INVERSES

Motivation: How can we find $\frac{d}{dx} f^{-1}(x)$ given $f(x)$ <u>w/out</u> explicitly calculating $f^{-1}(x)$ — which may be hard or impossible

Consider the (arbitrary) line $f(x) = mx + b$. (*)

Because (*) $\rightarrow$ $x = \frac{f(x)-b}{m}$ we have

$f^{-1}(x) = \frac{f(x)-b}{m} = \frac{1}{m} f(x) - \frac{b}{m}$ (another line).



So, geometrically speaking the tangent on the inverse function should exist.

We can make this relationship <u>algebraically</u> explicit using chain rule

By def^{n}

$$f(f^{-1}(x)) = x$$

$$\xRightarrow{\frac{d}{dx}} \frac{d}{dx} f(f^{-1}(x)) = \frac{dx}{dx}$$

we would like to calc this

$$\Rightarrow \frac{d}{dx} f \circ f^{-1}(x) \cdot \boxed{\frac{d}{dx} f^{-1}(x)} = 1$$

$$\Rightarrow \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

<u>Thm</u> Derivatives of Inverses

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{\frac{d}{dx} f(x) \circ f^{-1}(x)}$$

OR

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}.$$

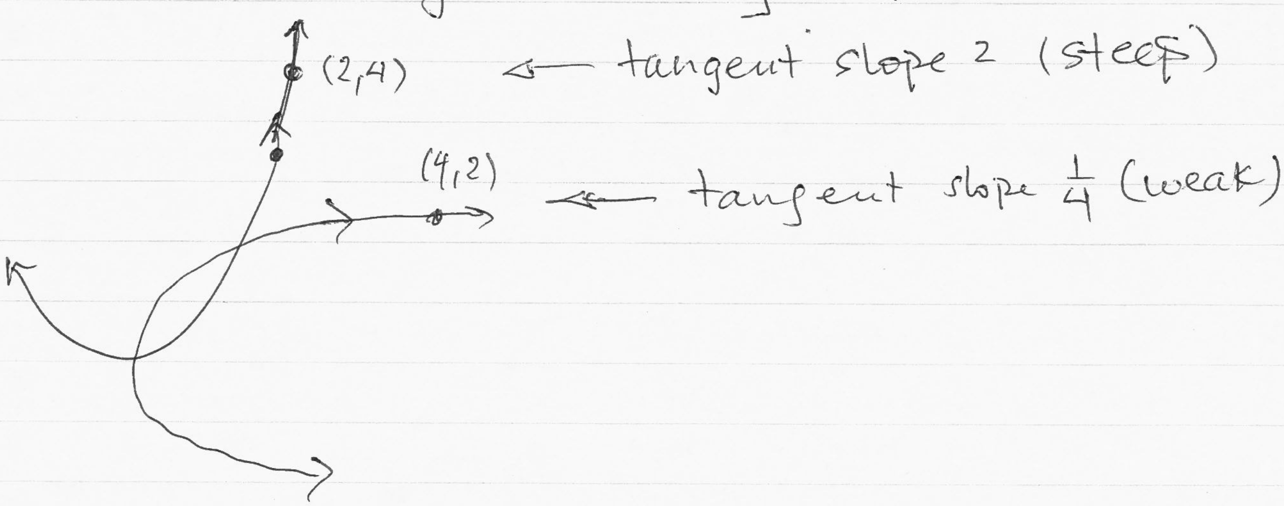
As $f^{-1}(x)$ is hard to find we normally use this Thm to calculate slopes at specific points.

<u>EXAMPLE</u> $f(x)=x^2$. Find $\frac{df^{-1}}{dx}(4)$

Note $(2,4) \in G(f(x)) \Rightarrow f(2)=4 \Rightarrow f^{-1}(4)=2$.

$$\frac{df^{-1}}{dx}(4) = \frac{1}{f'(x)\circ f^{-1}(4)} = \frac{1}{2x \circ 2} = \frac{1}{4}.$$

Is this sensible geometrically? YES!



EXAMPLE $f(x) = x^3 - 2$. Find $\frac{df^{-1}}{dx}(6)$

Note $f(2) = 8 - 2 = 6 \Rightarrow f^{-1}(6) = 2$ by guessing

$$\frac{df^{-1}}{dx}(6) = \frac{1}{f'(x) \circ f^{-1}(6)} = \frac{1}{3x^2 \circ 2} = \frac{1}{12}$$

(Confirm geometrically w/ desmos)

EXERCISE $f(x) = 2x^2 + 1$. Find $\frac{df^{-1}}{dx}(5) = \cdots = \frac{1}{8}$

Prop^n $\frac{d}{dx}\ln x = \frac{1}{x}$ $x > 0$

Proof: Let $f(x) = e^x \Rightarrow f^{-1}(x) = \ln x$

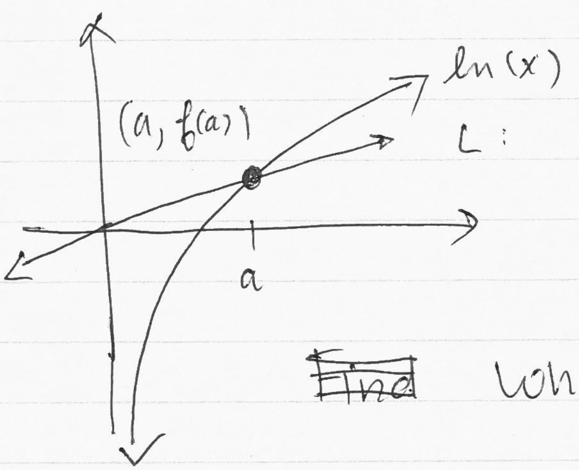
$$(f^{-1}(x))' = \frac{1}{f'(x) \circ \ln x} = \frac{1}{e^x \circ \ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}. \square$$

More generally: $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)} \cdot f'(x)$ $x > 0$.

Prop$^{\underline{n}}$ $\quad \dfrac{d}{dx} \ln|x| = \dfrac{1}{x} \qquad x \in \mathbb{R}.$

Proof $\quad \dfrac{d}{dx} \ln|x| = \dfrac{1}{|x|} \cdot (|x|)' = \dfrac{x}{|x|^2} = \dfrac{x}{(\sqrt{x^2})^2} = \dfrac{x}{x^2} = \dfrac{1}{x}$

EXAMPLE A line w/ slope m passes through the origin and is tangent to $y = \ln x$ — what is m?



$L: \quad y - \ln a = \dfrac{\ln a}{a}(x - a)$

$\Rightarrow$ L has slope $m = \dfrac{\ln a}{a}$

~~Find~~ Where does $\dfrac{\ln a}{a} = \dfrac{d \ln x}{dx}(a)$?

$\dfrac{d \ln x}{dx}(a) = \dfrac{1}{a}. \Rightarrow \dfrac{\ln a}{a} = \dfrac{1}{a} \Rightarrow \ln a = 1 \Rightarrow a = e.$

m is $\dfrac{\ln e}{e} = \dfrac{1}{e}$

Derivatives of $a^{f(x)}$ and $\log_a f$.

Propⁿ $\frac{d}{dx} a^{f(x)} = a^{f(x)} f'(x) \ln a$

Proof: $y = a^{f(x)}$. Find y'

NEW TECHNIQUE

$\Rightarrow \ln y = f(x) \ln a \overset{'}{\Rightarrow} \frac{1}{y} y' = f'(x) \ln a$

$\Rightarrow y' = y f'(x) \ln a \Rightarrow y' = a^{f(x)} f'(x) \ln a.$

(Remembering the technique is more important than the rule).

EXAMPLE

- $\frac{d}{dx} 3^{\sin x} = 3^{\sin x} \cdot \cos x \cdot \ln 3$

- $\frac{d}{dx} 2^x = 2^x \ln 2$

Prop$^{\underline{n}}$ $$\frac{d}{dx}\log_a f(x) = \frac{f'(x)}{\ln a \cdot f(x)}$$

Proof: $$y = \log_a f(x) = \frac{\ln f(x)}{\ln a}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x) \cdot \frac{1}{\ln a} \qquad \square$$

EXAMPLE Logarithmic Diff *New Technique*

Find $\frac{dy}{dx}$ when $$y = \frac{(x^2+1)\sqrt{x+3}}{x-1}$$

$$\Rightarrow \ln y = \ln(x^2+1) + \tfrac{1}{2}\ln(x+3) - \ln(x-1)$$

$$\Rightarrow \frac{1}{y}y' = \frac{1}{x^2+1}\cdot x^2 + \frac{1}{2(x+3)} - \frac{1}{(x-1)}$$

$$\Rightarrow y' = \frac{x^2 y}{x^2+1} + \frac{y}{2(x+3)} - \frac{y}{(x-1)}$$

<u>EXERCISE</u> let $y = x^x$ find y'

<u>EXERCISE</u> Find y' using logarithmic - diff.

- $y = \sqrt{x(x+1)}$

- $y = \dfrac{\theta \sin\theta}{(\sec\theta)^{\frac{1}{2}}}$

<u>EXERCISE</u> Find y'

- $y = \ln(3x) + x$

- $y = \ln(\sec(\ln\theta))$
- $y = t \ln\sqrt{t}$

§ Inverse Trig

Recall

$$\arcsin \sin\theta = \theta$$

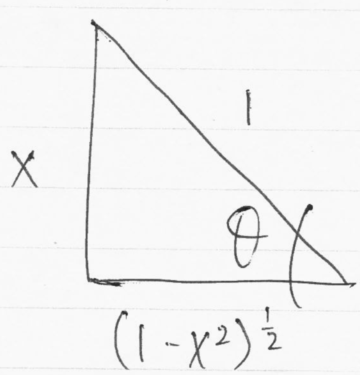
$$\arccos \cos\theta = \theta$$

$$\arctan \tan\theta = \theta$$

Prop^{n}: $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$ for $|x|<1$.

Proof: $(\arcsin x)' = \frac{1}{(\sin x)' \circ \arcsin x} = \frac{1}{\cos \arcsin x}$

Let $\theta = \arcsin \frac{x}{1}$



$$\Rightarrow \cos\theta = \cos \arcsin x = (1-x^2)^{\frac{1}{2}}$$

EXAMPLE $\frac{d}{dx} \arcsin x^2 = \frac{1}{(1-(x^2)^2)^{\frac{1}{2}}} \cdot (x^2)'$

$$= \frac{2x}{\sqrt{1-x^4}}$$

Prop$^{\underline{n}}$ $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

Proof:

$$\frac{d}{dx} \arctan x = \frac{1}{(\tan x)' \circ \arctan x}$$

$$= \frac{1}{\sec^2 \arctan x}$$

$$= \frac{1}{1 + \tan^2 \arctan x}$$

$*$ $\sec^2\theta = 1 + \tan^2\theta$

$$= \frac{1}{1+x^2}$$

$\square$

EXERCISE:
$$\frac{d}{dx} \arccos\theta$$

EXERCISE: Find y' given $y = \ldots$

A. $\arcsin\left(\frac{1}{\sqrt{2}}\right)$

B. $\operatorname{arcsec}(x^2)$

C. $\operatorname{arccot}\sqrt{t-1}$

D. $\ln \arctan x$

EXERCISE: Find

- $\lim_{x \to \infty} \arctan x$
- $\lim_{x \to \infty} \arcsin x$

$\vdots$

- $\lim_{x \to} \operatorname{arccsc} x$

all of the trig functions