§ CHAIN RULE

MOTIVATION : $\frac{d}{dx} \sin(x^2+2x) = ?$

Notice : $\sin(x^2+2x) = \sin x \circ (x^2+2x) = f \circ g$

for $f = \sin x$ and $g(x) = x^2 + 2x$.

If we can derive a rule for:

$$\frac{d}{dx}(f \circ g)$$

we're good-to-go.

<u>thm</u> Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

OR

$$(f \circ g)'(x) = f' \circ g(x) \cdot g'(x)$$

OR

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Proof: Omitted. But understandable.

EXAMPLE $y = (3x^2+1)^2$

- let $y = u^2$ w/ $u = 3x^2+1$. Chain Rule $\Rightarrow$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{w/} \quad \frac{dy}{du} = 2 \cdot u = 2(3x^2+1)$$

and $\frac{du}{dx} = 6x$ $\Rightarrow$ $\frac{dy}{dx} = 2 \cdot (3x^2+1) \cdot (6x)$

$$= 36x^3 + 12x$$

~OR~

- $y = x^2 \circ (3x^2+1)$

$$\Rightarrow \frac{dy}{dx} = (x^2)' \circ (3x^2+1) \cdot (3x^2+1)'$$

$$= 2x \circ (3x^2+1) \cdot (6x)$$

$$= 2(3x^2+1) \cdot 6x = 36x^3 + 12x.$$

(Either way is fine).

EXAMPLE Let $x(t) = \cos(t^2+1)$. Find velocity.

$$x(t) = \cos t \circ (t^2+1)$$

$$\Rightarrow \frac{dx}{dt} = (\cos t)' \circ (t^2+1) \cdot (t^2+1)'$$

$$= -\sin t \circ (t^2+1) \cdot (2t)$$

$$= -\sin(t^2+1) \cdot 2t$$

EXAMPLE $y = \sin(x^2+e^x)$

$$y = \sin x \circ (x^2+e^x)$$

$$\rightarrow \frac{dy}{dx} = (\sin x)' \circ (x^2+e^x) \cdot (x^2+e^x)'$$

$$= \cos x \circ (x^2+e^x) \cdot (2x+e^x)$$

$$= \cos(x^2+e^x) \cdot (2x+e^x)$$

EXAMPLE $y = e^{\cos x} \rightarrow y = e^x \circ \cos x$

$$\Rightarrow \frac{dy}{dx} = (e^x)' \circ \cos x \cdot (\cos x)'$$

$$= e^x \circ \cos x \cdot (-\sin x)$$

$$= -e^{\cos x} \cdot \sin x$$

EXAMPLE: $g(t) = \tan(5-\sin 2t)$

$$\Rightarrow g(t) = \tan t \circ (5 - \sin t \circ 2t)$$

$$\Rightarrow \frac{dg}{dt} = (\tan t)' \circ (5-\sin t \circ 2t) \cdot (5-\sin t \circ 2t)'$$

$$= \sec^2 t \circ (5-\sin t \circ 2t) \cdot (0 - \cos t \circ 2t \cdot (2t)')$$

$$= \sec^2(5-\sin 2t)(-\cos 2t \cdot 2)$$

Eventually we must be able to do this w/out all the detail:

$$\frac{dg}{dt} = \sec^2(5-\sin 2t) \cdot (5-\sin 2t)'$$

$$= \sec^2(5-\sin 2t) \cdot (-\cos 2t \cdot 2).$$

EXAMPLE: $y = (5x^3 - x^4)^7$

$$\Rightarrow y = x^7 \circ (5x^3 - x^4)$$

$$\Rightarrow \frac{dy}{dx} = (x^7)' \circ (5x^3 - x^4) \cdot (5x^3 - x^4)'$$

$$= 7x^6 \circ (5x^3 - x^4) \cdot (15x^2 - 4x^3)$$

$$= 7(5x^3 - x^4)^6 \cdot (15x^2 - 4x^3)$$

<u>EXAMPLE:</u> $y = \dfrac{1}{3x-2}$

$\Rightarrow y = (3x-2)^{-1} \Rightarrow \dfrac{dy}{dx} = -(3x-2)^{-2} \cdot (3x-2)'$

$= \dfrac{-3}{(3x-2)^2}$

<u>EXAMPLE:</u> $y = e^{\sqrt{3x+1}} = \exp\left((3x+1)^{\frac{1}{2}} \right)$

Notation: $\exp(x) = e^x$

Here $y = \exp(x) \circ x^{\frac{1}{2}} \circ (3x+1)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{d}{dx} \exp(x) \circ x^{\frac{1}{2}} \circ (3x+1) \cdot \dfrac{d}{dx}\left(x^{\frac{1}{2}} \circ (3x+1) \right)$

writing like this is messy... Better:

$\dfrac{dy}{dx} = \exp\left((3x+1)^{\frac{1}{2}} \right) \cdot \dfrac{d}{dx} (3x+1)^{\frac{1}{2}}$

$= \exp\left((3x+1)^{\frac{1}{2}} \right) \cdot \frac{1}{2} (3x+1)^{-\frac{1}{2}} \cdot \dfrac{d}{dx}(3x+1)$

$= \exp\left((3x+1)^{\frac{1}{2}} \right) \cdot \dfrac{3}{2(3x+1)^{\frac{1}{2}}}$

$= e^{\sqrt{3x+1}} \cdot \dfrac{3}{2\sqrt{3x+1}}$.

<u>EXAMPLE:</u> $y = \sin^5 x = (\sin x)^5$

$\Rightarrow \frac{dy}{dx} = 5(\sin x)^4 \cdot (\sin x)'$

$= 5 \sin^4 x \cdot \cos x$

<u>EXAMPLE</u>: $y = |x| = \sqrt{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{d\,(x^2)^{\frac{1}{2}}}{dx} = \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot (x^2)'$

$= \frac{2x}{2\sqrt{x^2}} = \frac{x}{|x|}$. for $x \neq 0$.

<u>EXAMPLE:</u> Show the slope of all tangent lines to $y = \frac{1}{(1-2x)^3}$ is positive.

EQ: Show $dy/dx > 0$. for all x.

$y = (1-2x)^{-3} \rightarrow \frac{dy}{dx} = -3(1-2x)^{-4} \cdot (1-2x)'$

$= \frac{6}{(1-2x)^4}$,

Notice $(1-2x)^4 \geq 0 \Rightarrow \frac{1}{(1-2x)^4} \geq 0 \rightarrow \frac{6}{(1-2x)^4} \geq 0$

$\Rightarrow \frac{dy}{dx} \geq 0$.

EXERCISE: Let $f'(3) = -1$ $\quad g(2) = 3$

$g'(2) = 5 \quad y = f(g(x))$.

What is $y'(2)$?

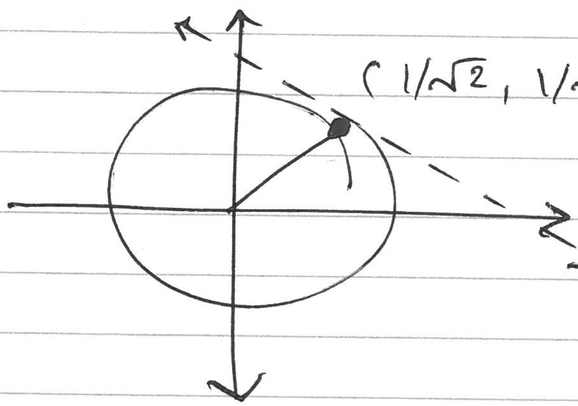
EXERCISE $\quad y = \left(\dfrac{t^2}{t^3 - 4t}\right)^3 \Rightarrow \dfrac{dy}{dt} = ?$

EXERCISE Expand $\dfrac{d}{dx}\left(f(g(h(x)))\right)$.

§ Implicit Differentiation

Motivation: How can we differentiate eqns we <u>cannot</u> write as functions?

Eg. What is the slope of the tangent on the <u>unit circle</u> at $p = (1/\sqrt{2}, 1/\sqrt{2})$?



Find $\left.\frac{dy}{dx}\right|_{(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

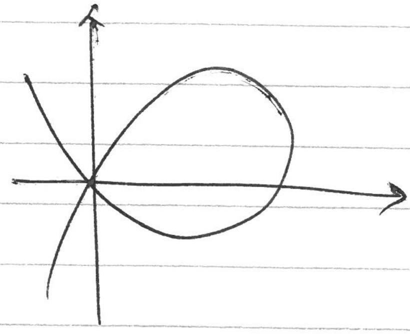
Circle: $x^2 + y^2 = 1$

$\Rightarrow 2x \cdot \frac{dx}{dx} + 2y \cdot \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$ (notice x & y)

$\Rightarrow \left.\frac{dy}{dx}\right|_{(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1.$

EXAMPLE Investigate derivatives of



$$x^3 + y^3 - 9xy = 0 \quad (*)$$

at the origin.

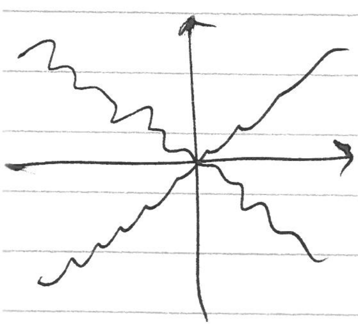
$$\Rightarrow \frac{d(*)}{dx}: \quad 3x^2 \cdot \frac{dx}{dx} + 3y^2 \cdot \frac{dy}{dx} - 9\frac{dx}{dx}y - 9x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3x^2 + \frac{dy}{dx}(3y^2 - 9x) - 9y = 0$$

$$\rightarrow \quad \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

will not give you slope at origin — because there are two.

EXAMPLE: Find $\frac{dy}{dx}$ when $y^2 = x^2 + \sin xy \quad (*)$



$$\frac{d(*)}{dx}: \quad 2y \cdot \frac{dy}{dx} = 2x \cdot \frac{dx}{dx} + \cos(xy) \cdot \left(\frac{dx}{dx}y + x\frac{dy}{dx}\right)$$

and isolate for $\frac{dy}{dx}$.

EXAMPLE: let $xy + y^2 = 1$, find y''

Let A: $xy + y^2 = 1$

$$\Rightarrow \frac{dA}{dx}: \quad x'y + xy' + 2yy' = 0$$

$$\Rightarrow \quad y + xy' + 2yy' = 0 \qquad : B$$

$$\Rightarrow \quad y'(x+2y) = -y$$

$$\Rightarrow \quad y' = \frac{-y}{(x+2y)}$$

Proceed w/ quotient rule or...

$$\frac{dB}{dx}: \quad y' + x'y' + xy'' + 2y'y' + 2yy'' = 0$$

$$\Rightarrow \quad y''(x+2y) + 2y' + 2(y')^2 = 0$$

$$\Rightarrow \quad y'' = -(2y' + 2(y')^2)/(x+2y)$$

$$= -\left(\frac{-2y}{(x+2y)} + 2\left(\frac{-y}{x+2y}\right)^2\right)/(x+2y)$$

$$= \frac{2y}{(x+2y)^2} - \frac{2y^2}{(x+2y)^3}.$$

EXERCISE Find slope of the tangent on

$(x^2+y^2)^2 = (x-y)^2$ at $(1,0)$ and $(1,-1)$.

EXERCISE Find $\frac{dr}{d\theta}$ when $\cos r + \cot\theta = e^{r\theta}$.

EXERCISE Find $\frac{dy}{dx}$ when $x^2y + xy^2 = 6$.

EXERCISE Find two points where

$$x^2 + xy + y^2 = 7.$$

crosses the x-axis. and show the tangents to the curve at those points are parallel.

EXERCISE The line that is normal to the curve

$$x^2 + 2xy - 3y^2 = 0$$

at $(1,1)$ intersects the curve at what other point?