§DERIVATIVES OF TRIG FUNCTIONS

MOTIVATION: What is $\frac{d}{dx}\sin x$?

By definition...

$$\frac{d}{dx}\sin x = (\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

STUCK $\lim_{h\to 0}\frac{\cos h - 1}{h} = ?$ $\lim_{h\to 0}\frac{\sin h}{h} = ?$

Propⁿ $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

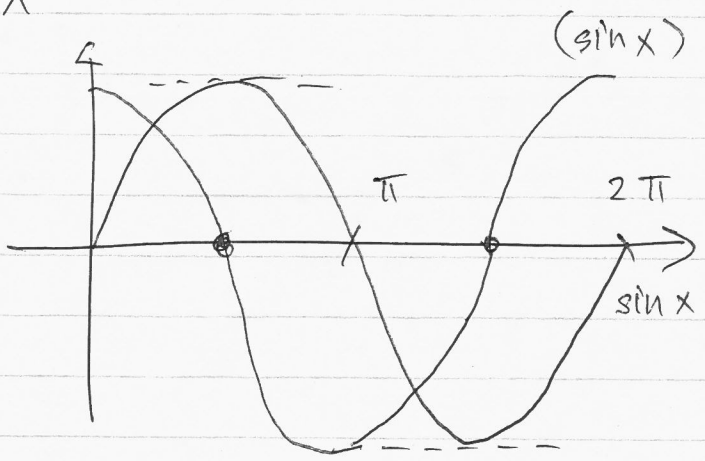
Proof: Omitted.

Propⁿ $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$

Proof $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0} -\frac{2\sin^2(\theta/2)}{\theta}$

Let $h = \theta/2$ .. $= \lim_{h\to 0} -\frac{\sin^2(h)}{h} = \lim_{h\to 0} -\sin h\cdot\frac{\sin h}{h} = 0.$

Does $\frac{d}{dx}\sin x = \cos x$ make sense geometrically?



Propⁿ $\frac{d}{dx}\cos x = \frac{1}{\cos^2 x} = \sec^2 x$

Proof: Basically the same as $\sin x$.

Propⁿ $\frac{d}{dx}\tan x$.

Proof: $\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{(\sin x)'\cos x - \sin x\,(\cos x)'}{\cos^2 x}$

$= \frac{\cos x \cdot \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$ □

EXAMPLE $y = 5e^x + \cos x \Rightarrow y' = 5e^x - \sin x$

EXAMPLE $y = \sin x \cos x$

$$\Rightarrow y' = (\sin x)' \cos x + \sin x (\cos x)'$$

$$= \cos^2 x - \sin^2 x$$

EXAMPLE $y = \dfrac{\cos x}{1 - \sin x}$

$$\Rightarrow y' = \frac{-\sin x (1 - \sin x) - \cos x (-\cos x)}{(1 - \sin x)^2}$$

$$= \frac{\sin^2 x + \cos^2 x - \sin x}{(1 - \sin x)^2} = \frac{1 - \sin x}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

EXAMPLE $y = \sec x$

$$\Rightarrow y' = \left(\frac{1}{\cos x}\right)' = \frac{0 \cdot \cos x - 1 \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \tan x.$$

EXAMPLE $y = \sec x$ Cont...

$$y'' = \left(\frac{1}{\cos x}\tan x\right)'$$

$$= \left(\frac{1}{\cos x}\right)'\tan x + \frac{1}{\cos x}(\tan x)'$$

$$= \frac{\tan^2 x}{\cos x} + \frac{1}{\cos^3 x}$$

$$= \sec x \tan^2 x + \sec^3 x$$

either are fine

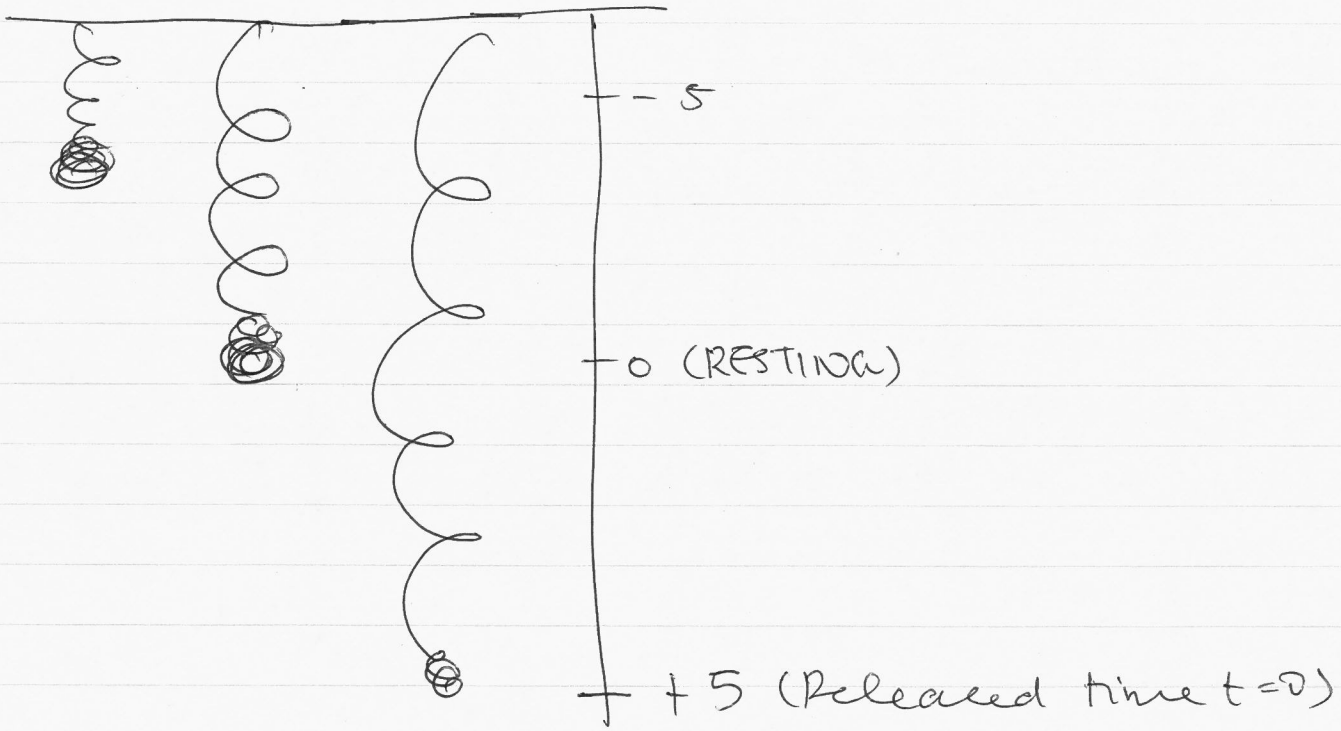
EXERCISE

$$\lim_{x \to 0} \frac{\sin(3x)}{2x}$$

EXERCISE Where does $\cos x$ have tangent slope $\sqrt{2}$?

EXERCISE A weight from a spring is extended 5 units from rest position and released at time $t=0$. Its position at any time is given by:

$$s(t) = 5\cos t$$



(1.) Find velocity and acceleration ($v(t)$ & $a(t)$).

(2.) When is the ball moving fastest?

(3.) Asides at rest, when else does the spring have zero acceleration?