§ Differentiation Rules

We dont want to take limits every time we need a derivative. Its better to have some alg. <u>rules</u> instead.

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

and $\frac{d}{dx}$: functions $\to$ functions
$f \mapsto f'$

We usually ~~interchange~~ dont distinguish $f'(x)$ and $\frac{df(x)}{dx}$ but technically one is "geometric" (uses limits) and the other "algebraic" (uses rules).

So let's establish some rules...

<u>Propⁿ</u> Let $c \in \mathbb{R}$

$$\frac{d}{dx} c = 0$$

<u>Proof:</u> $\frac{d}{dx} c = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ w/ $f(x) = c$

$$= \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0.$$

Propⁿ = "Rule"

Propⁿ $\frac{d}{dx} x^n = n x^{n-1}$

Proof: Let $f(x) = x^n$: $n \in \mathbb{N}$, $n > 0$

$$\frac{d}{dx} x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = (*)$$

STUCK: How can we expand $(x+h)^n$?

Binomial Thm (You dont need to know this)

Let $\binom{n}{k}$ = number of ways you can choose k-objects from n-objects.

$$= \frac{n!}{(n-k)!\,k!}$$ then

$$(a+b)^n = \binom{n}{0} a^{n-0} y^0 + \binom{n}{1} a^{n-1} y^1 + \cdots$$

$$+ \binom{n}{n-1} a^1 y^{n-1} + \binom{n}{n} a^0 y^{n-0}$$

So here

$$(x+h)^n = x^n + n x^{n-1} h + \binom{2}{n} x^{n-2} h^2 + \cdots + h^n$$

$(x+h)^n$

Thus $\circledast = \lim_{h \to 0} \dfrac{(x^n + nx^{n-1}h + \binom{2}{n} x^{n-2}h^2 + \cdots + h^n) - x^n}{h}$

$= \lim_{h \to 0} \dfrac{nx^{n-1}h + \binom{2}{n} x^{n-2}h^2 + \cdots + h^n}{h}$

$= \lim_{h \to 0} \underset{\uparrow \text{ no } h}{nx^{n-1}} + \binom{2}{n} \underbrace{x^{n-2}h + \cdots + h^{n-1}}_{\text{every term has } h.}$

$= nx^{n-1} + 0 + \cdots + 0$

$= nx^{n-1}.$ $\square$

POWER RULE $\dfrac{d}{dx} x^n = nx^{n-1}$ for $n \in \mathbb{R}$.

Proof: Outside scope (i.e. too hard)

EXAMPLE

- $\dfrac{dx^3}{dx} = 3x^{3-1} = 3x^2$
- $\dfrac{d\,3x^7}{dx} =$ Requires another rule!
- $\dfrac{d\,x^{\sqrt{2}}}{dx} = \sqrt{2}x^{\sqrt{2}-1}$

What about $\frac{d}{dx}(x^{2/3} + x^2)$?

We need...

SUM RULE $\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g$

Proof: Omitted. But we have the tools to do it.

Now we can say

EXAMPLE $\frac{d}{dx}(x^{2/3} + x^2)$

$= \frac{d}{dx}x^{2/3} + \frac{d}{dx}x^2$ SUM RULE

$= \frac{2}{3}x^{2/3-1} + 2x^1$ POWER RULE

Prop^n $\frac{d}{dx}cx^n = c \cdot n x^{n-1}$

Proof: $\frac{d}{dx}cx^n = \frac{d}{dx}(\underbrace{x^n + x^n + \cdots + x^n}_{c\text{ times}})$

$= \frac{d}{dx}x^n + \cdots + \frac{d}{dx}x^n$

$= nx^{n-1} + \cdots + nx^{n-1}$

$= cn \cdot x^{n-1}$ $\square$.

*just writing 0 for emphasis today only
don't write zero*.

Notice we have enough rules ~~powers~~ to take derivatives of arbitrary polynomials : $a_n x^n + \dots + a_1 x + a_0$

EXAMPLE :

$$\frac{d}{dx}(3x^2 - 2x + 1)$$
$$= 6x - 2 + 0 \qquad *$$

EXAMPLE Where does $y = x^4 - 2x^2 + 2$ have horizontal tangent lines?

EQUIVALENT: Where is $\frac{dy}{dx} = f'(x) = 0$?

$$\frac{dy}{dx} = 4x^3 - 4x + 0 \quad \text{and} \qquad *$$

$$0 = 4x^3 - 4x \Rightarrow 4x^3 = 4x \Rightarrow x^2 = 1 \Rightarrow x = 1, -1.$$

WRONG! We divided by hidden ~~zero~~. $x = 0$.

CORRECT $4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$

So $x = 0, 1, -1. \Rightarrow y$ has a ~~h~~ horizontal tangent.

Prop^n $$\boxed{\frac{d}{dx} e^x = e^x}$$

Proof: Too long. But we could do it.

How about $(x^2 \cdot e^x)'$?

PRODUCT RULE

$$\frac{d}{dx}(f \cdot g) = \left(\frac{df}{dx}\right) g + f\left(\frac{dg}{dx}\right)$$

OR

$$(f \cdot g)' = f'g + fg'$$

Proof: EXERCISE (for fun)

EXAMPLE $y = \frac{1}{x}(x^2 + e^x)$

$$\Rightarrow y' = \left(\frac{1}{x}\right)'(x^2 + e^x) + \left(\frac{1}{x}\right)(x^2 + e^x)'$$

$$= (x^{-1})'(x^2 + e^x) + (x^{-1})(x^2 + e^x)'$$

$$= (-x^{-2})(x^2 + e^x) + (x^{-1})(2x + e^x)$$

$$= -1 - e^x/x^2 + 2 + e^x/x$$

$$= 1 - e^x/x^2 + e^x/x$$

EXAMPLE $y = (x^2+1)(x^3+3)$

$$y' = (2x)(x^3+3) + (x^2+1)(3x^2)$$

$$= 2x^4 + 6x + 3x^4 + 3x^2$$

$$= 5x^4 + 3x^2 + 6x$$

QUOTIENT RULE

$$\left(f/g\right)' = \frac{f'g - fg'}{g^2} \quad (*)$$

Proof Once we have chain rule we can easily demonstrate $(*)$ by applying product rule to $\left(f \cdot \frac{1}{g}\right)'$.

However, geometric/limit defn will work as well.

EXAMPLE $y = \dfrac{(x-1)(x^2-2x)}{x^4}$

$$y' = \frac{((x-1)(x^2-2x))'x^4 - ((x-1)(x^2-2x))(x^4)'}{(x^4)^2}$$

$$= \frac{\left((x-1)'(x^2-2x) + (x-1)(x^2-2x)'\right)x^4 - ((x-1)(x^2-2x))(4x^3)}{x^8} = \ldots$$

EXERCISE.

OR we can do

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4}$$

$$\Rightarrow y' = \frac{(x^3-3x^2+2x)'(x^4) - (x^3-3x^2+2x)(x^4)'}{(x^4)^2}$$

$$= \frac{(3x^2-6x+2)(x^4) - (\cancel{3}x^3-3x^2+2x)(4x^3)}{x^8}$$

= EXERCISE - - -

You can take a derivative twice — this is called the 2^{nd} derivative

<u>EXAMPLE</u>

$$(x^4)'' = ((x^4)')' = (4x^3)' = 12x^2.$$

§ Rates of Change

Calculus models basic Newtonian physics.

For instance, suppose (roughly speaking) we could measure a particles position ~~or displacement~~

Then velocity = how fast position is changing

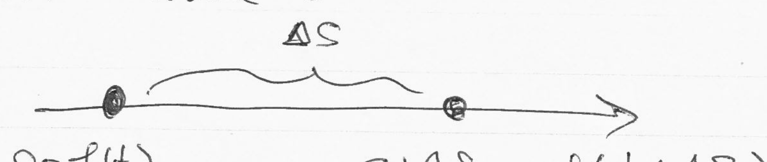
acceleration = how fast velocity is changing.

§ Displacement

We encode a particle's position by <u>displacement</u> from some initial position.

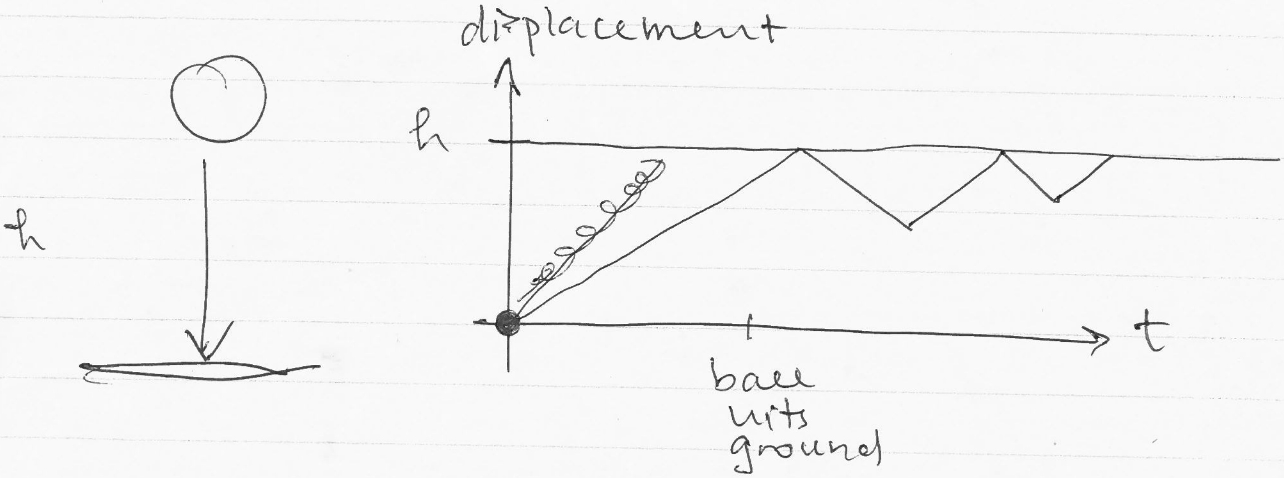
This is <u>not</u> distance travelled. If I drive to Florida <u>and back</u> I have zero displacement but traversed a big distance.

<u>EXAMPLE</u> $f(t)$ gives position on a number line at time t.



Particle is <u>displaced</u> by Δs from t to $t+\Delta t$ seconds.

EXAMPLE Draw displacement v. time for a bouncing ball dropped from height h.



In reality the lines would be curved.

Def^n Displacement

Let $s = f(t)$ be the position of a particle along a real line at time t. The displacement of the particle over t to $t + \Delta t$ is given by

$$\Delta s = f(t+\Delta t) - f(t)$$

§ Velocity

Velocity measures how rapidly displacement is changing.

~~First consider "average" than "instantaneous"~~

Def$^{\underline{n}}$ Velocity (Average)

$$V_{av} = \frac{\Delta s}{\Delta t} = \frac{\text{displacement}}{\text{travel time}}$$

$$= \frac{f(t+\Delta t) - f(t)}{\Delta t}$$ (weirdly similar to tangent---)

EXAMPLE It ~~took~~ Alice 6-hours to drive ~~so~~mewhere 10km away

~~After the~~ Thus her average velocity was:

$$V_{av} = \frac{12\,Km}{6\,Hrs} = 2Km/hr$$

This does not mean Alice never stopped, sped up, or slowed down. Just that on average Alice was driving 2 Km/hr.

QUESTION How fast was ~~alice~~ Alice travelling 1 hour into her trip?

We need instantaneous velocity. — velocity at an instant in time.

<u>Defn</u> Velocity (Instantaneous)

$$V(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Note similarity w/ ave: $\dfrac{\Delta s}{\Delta t} \xrightarrow{+\text{ limits}} \dfrac{ds}{dt}$

Instantaneous velocity enables us to answer "how fast is the displacement changing at t."

<u>EXAMPLE</u> Given $s(t) = t^2 + 3$ is a particles displ. at time t (maybe a rocket ship blasted into space never to return...)

What is the ~~displacement~~ velocity of the particle at $t = 3s$?

$$V(t) = \frac{ds}{dt} = \frac{d(t^2+3)}{dt} = 2t + 0$$

$$\Rightarrow V(3) = 6\, m/s.$$

<u>Defn</u> Speed

$$\text{Speed} = |V(t)| = \left|\frac{ds}{dt}\right|$$

§ Acceleration

Acceleration measures how rapidly the velocity is changing

Def^n Acceleration

let $v(t)$ give ~~displ~~ velocity and $s(t)$ displacement. Then

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

2nd derivative

gives acceleration.

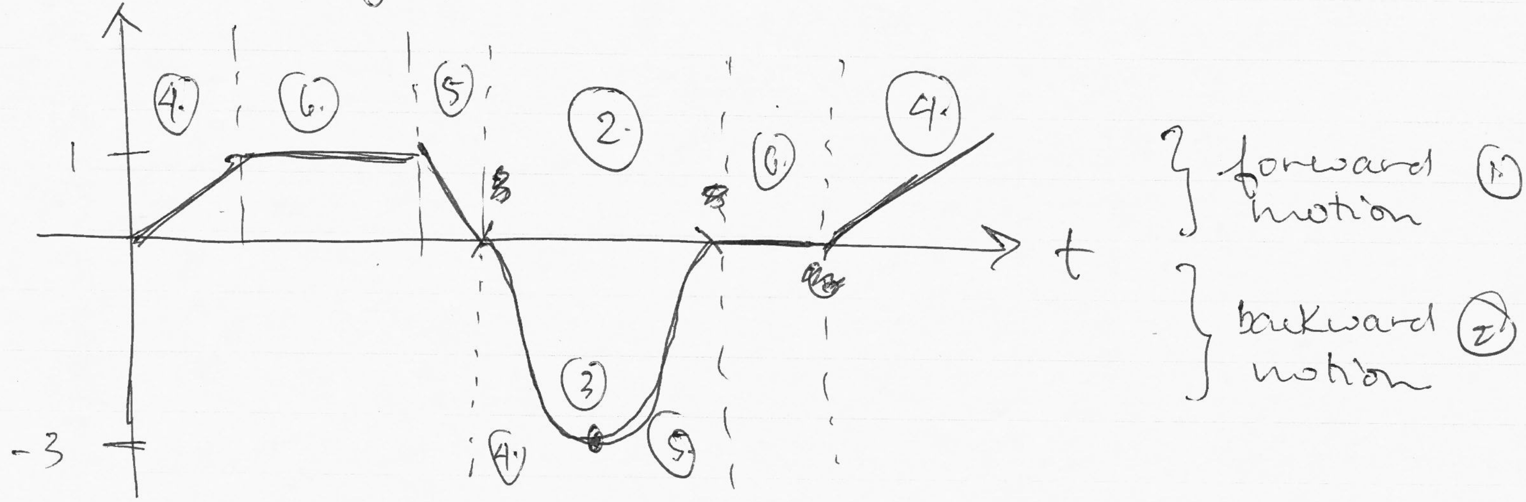
EXAMPLE $s(t) = t^2 + 3 \Rightarrow v(t) = 2t \text{ m/s} \Rightarrow a(t) = 2 \text{ m/s}^2$

From $t=0$ (i.e. rest) after 1 second velocity is 2, at $t=1$ $v=4$, and $t=2$ $v=6$.

Notice $a(t)$ says velocity increases by 2m/s every second

EXERCISE Interpret this velocity graph.

v(t) – velocity at time t.



1. Where is part. moving:

1. forward
2. backward
3. "fastest" (i.e. greatest speed)
4. speeding up
5. slowing down
6. steady

EXERCISE

An object in free-fall has displacement

$$s(t) = 5t^2 \qquad \text{(approximately)}$$

(a) How ~~man~~ far does ~~the~~ an object fall in 3 seconds?

(b) What is the velocity, speed, and acceleration of the object at 3 seconds?