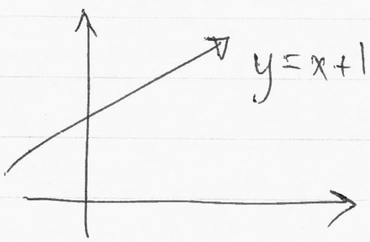
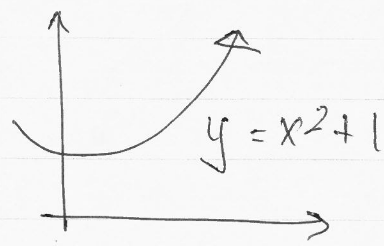
§ Tangents & Derivatives

MOTIVATION:

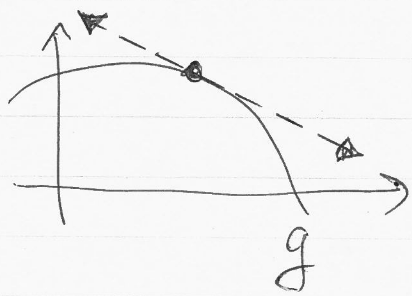
1. We need a way to measure how quickly a function is increasing.



increasing more slowly than

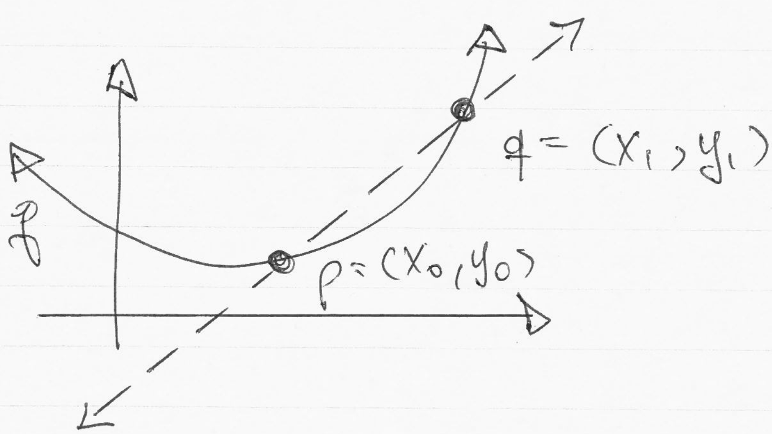


2. Curves look like straight lines when looked at closely



g is approximated by its "tangent".

Def$^{\underline{n}}$ Secant Line



The line connecting two points $p, q \in G(f)$ is called the "secant" through p & q.

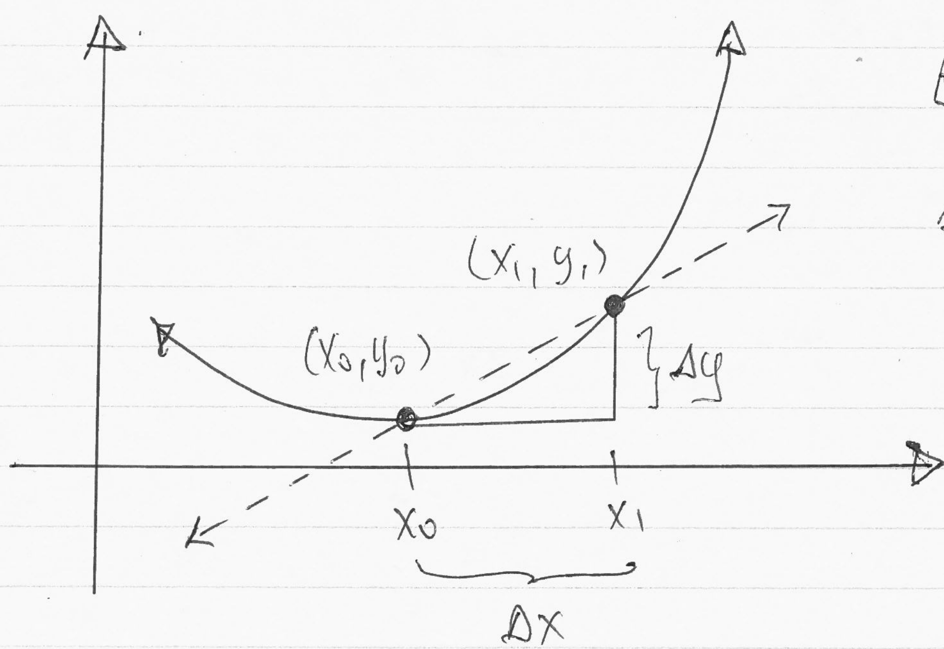
Algebraically the secant line L is given by:

$$L: \; y - y_0 = \underbrace{\left(\frac{y_1 - y_0}{x_1 - x_0}\right)}_{\text{slope of the secant}}(x - x_0)$$

or also equivalently

$$y - f(x_0) = \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x_1 - x_0).$$

Geometrically:



$\Delta y = y_1 - y_0$

$\Delta x = x_1 - x_0$

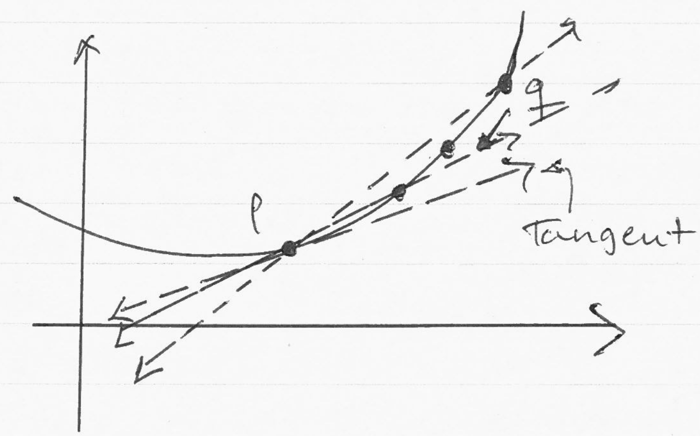
If we let $x_1 \to x_0$ (ie. $\Delta x \to 0$) we get the "tangent line".

Def^n Tangent Line

~~The "tangent line" of f(x)~~

Let $\overline{pq}$ denot the secant througe p & q on $f(x)$. The "tangent line" of $f(x)$ is given by

$$\lim_{p \to q} \overline{pq}.$$



Finding the tangent line requires calculating

$$\lim_{x_1 \to x_0} y - y_0 = m(x - x_0)$$

where $m = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)$.

$$\lim_{x \to x_0} \left(\frac{y_1 - y_0}{x_1 - x_0}\right) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

We can move x_1 into x_0 by letting $x_1 = x_0 + h$ and taking $h \to 0$.

$*$ $:=$ means defined by

Def^n Derivative

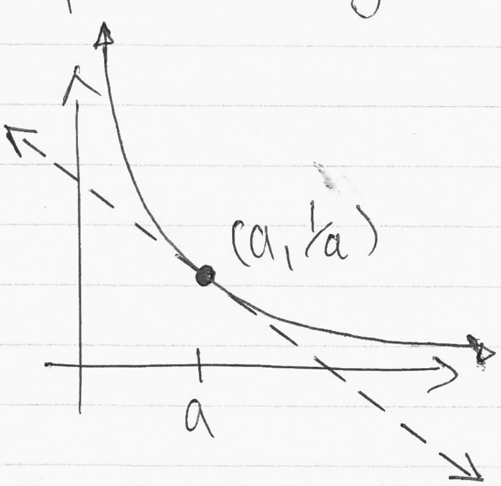
The derivative of f at $x \in \text{dom} f$ is denoted $f'(x)$ and

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(This is equivalent to finding the slope of the tangent of $f(x)$ at p).

EXAMPLE What is the slope, say m, of the tangent line of $f(x) = \frac{1}{x}$ at $x = a > 0$.

Equivalently: What is $f'(a)$?



$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \to 0} \frac{a - (a+h)}{h(a+h)(a)}$$

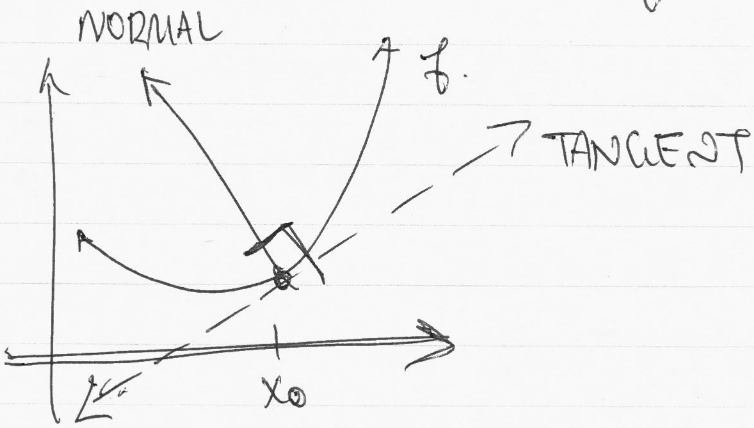
$$= \lim_{h \to 0} \frac{-h}{h(a+h)a} = \lim_{h \to 0} \frac{-1}{(a+h)a}$$

$$= \frac{-1}{(a+0)a} = -1/a^2 \Leftrightarrow f'(a) = \frac{-1}{a^2}$$

Def^{n} Normal

The normal line to a function f ~~at $x_0 \in$ dom f~~ at $x_0 \in \text{dom} f$ is the line perpendicular to the tangent line there. It is given by:

$$l: \; y - f(x_0) = \frac{-1}{f'(x_0)}(x - x_0)$$



EXAMPLE The normal at $x = a$ on $f(x) = 1/x$ is

$$y - \frac{1}{a} = \frac{-1}{\left(\frac{-1}{a^2}\right)}(x - a) \Rightarrow y - \frac{1}{a} = a^2(x - a)$$

§ Derivative of a function

The calculation of a function's derivative is called "differentiation".

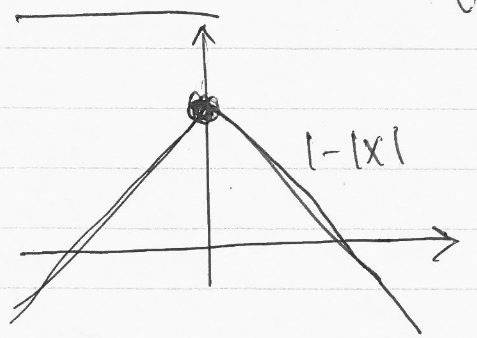
Def^n Differential Operator

$$\frac{d}{dx}: \text{functions} \longrightarrow \text{functions}$$
$$f(x) \longmapsto f'(x)$$

EXAMPLE $\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$

NOTE $\frac{df(x)}{dx} = \frac{d}{dx}f(x) = f'(x)$

EXAMPLE Let $f(x) = 1-|x|$. What is $f'(0)$?



Notice: $\lim_{h \to 0^+} \frac{(1-|0+h|)-(1-|0|)}{h}$

$= \lim_{h \to 0^+} \frac{(1-(0+h))-(1-0)}{h}$ because $x > 0$

$= \lim_{h \to 0^+} -h/h = \lim_{h \to 0^+} -1 = -1$

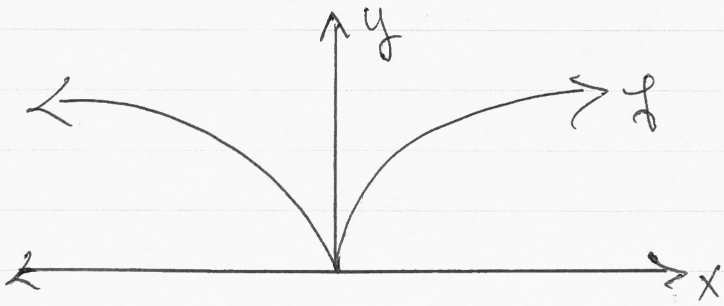
but $\lim\limits_{h \to 0^-} \dfrac{(1-|0+h|)-(1-|0|)}{h}$

$= \lim\limits_{h \to 0^-} \dfrac{(1-(0-h))-(1-0))}{h}$ because $h<0$

$= \lim\limits_{h \to 0^-} h/h = 1$

Thus $\lim\limits_{h \to 0} 1-|x|$ DNE and so neither does ~~the limit~~ $f'(x)$.

EXAMPLE Let $f(x) = x^{2/3}$. What is $f'(0)$?

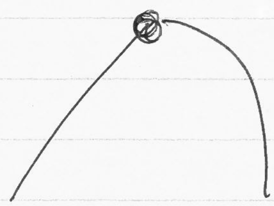


$f(x)$ does not have a tangent/derivative at $x=0$ because the slope of the tangent is $\pm\infty$.

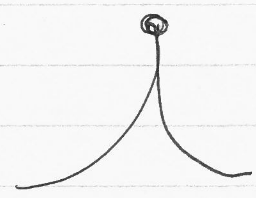
$f'(0) = \lim\limits_{h \to 0^+} \dfrac{(0+h)^{2/3} - 0^{2/3}}{h} = \lim\limits_{h \to 0^+} \dfrac{h^{2/3}}{h} = \lim\limits_{h \to 0^+} \dfrac{1}{h^{1/3}}$

$= \left(\lim\limits_{h \to 0^+} \dfrac{1}{h}\right)^{\frac{1}{3}} = +\infty$

Places where the derivative is undefined:



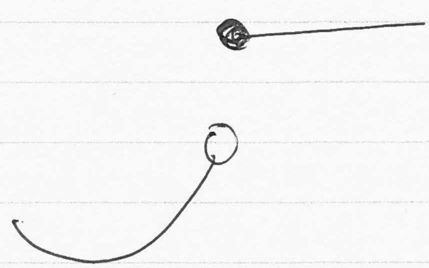
"CORNER"



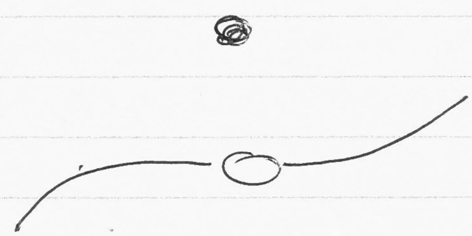
"CUSP"

There tangent slope is $+\infty$ from one direction and $-\infty$ from the other.

"VERTICAL TANGENT"



DISCONTINUITY: JUMP



REMOVEABLE DISCONTINUITY.