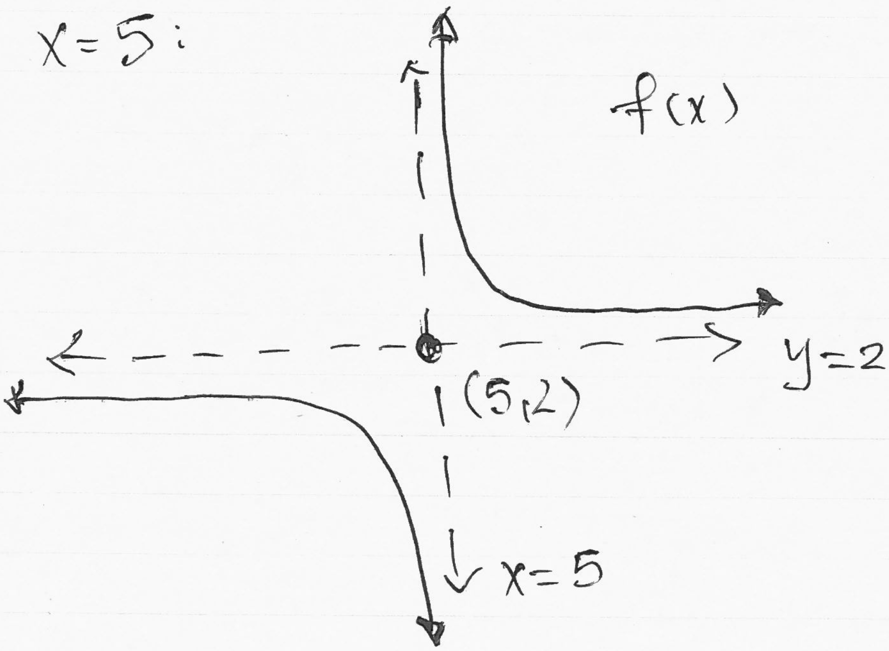
§LIMITS INVOLVING INFINITY

We have already seen vertical asymptotes which occur when

$$\lim_{x \to a^+} f(x) = +\infty, -\infty$$

$$\lim_{x \to a^-} f(x) = +\infty, -\infty$$

as in $f(x) = \frac{1}{x-5} + 2$ which has an asymptote at $x = 5$:



Notice there is another asymptote at $y=2$ — a horizontal asymptote.

* We use a strategy: ~~divid~~ isolate dominant term.

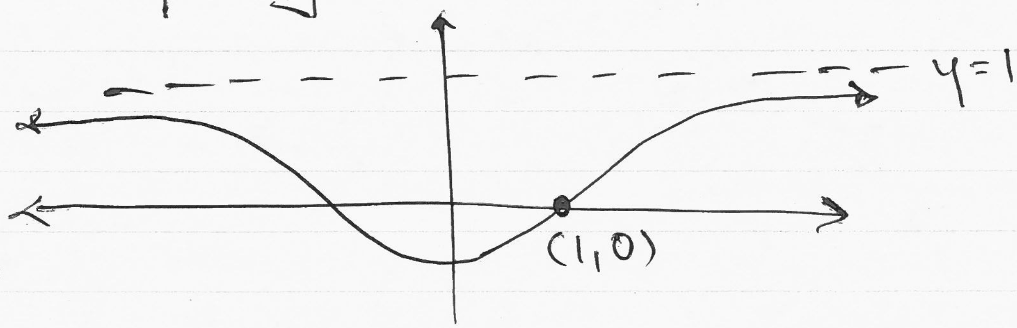
Def$^{\underline{n}}$ Horizontal Asymptote (HA)

$f(x)$ has the horizontal asymptote $y=b$ when

$$\lim_{x\to\infty} f(x) = b \quad \underline{\text{OR}} \quad \lim_{x\to-\infty} f(x) = b.$$

Note a function ~~have~~ can have up to two HAs — one for each infinity: $+\infty$, $-\infty$.

EXAMPLE



$$y = \frac{x^2-1}{x^2+1} = f(x)$$

Geometry suggests $\lim_{x\to\infty} f(x) = 1$.

Algebraically (using limit laws) *

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to\infty} \frac{\frac{1}{x^2}(x^2-1)}{\frac{1}{x^2}(x^2+1)} = \lim_{x\to\infty} \frac{\left(\frac{x^2}{x^2} - \frac{1}{x^2}\right)}{\left(\frac{x^2}{x^2} + \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 1 + \underbrace{\lim_{x\to\infty} \frac{1}{x^2}}_{\text{we have no rule for this}}} = *$$

STUCK

Prop$^{\underline{n}}$ $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to -\infty} \frac{1}{x} = 0$

Proof: Requires strict defn — omitted for MAT134.

Thus $\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} (\frac{1}{x})(\frac{1}{x}) = \lim_{x\to\infty} \frac{1}{x} \lim_{x\to\infty} \frac{1}{x}$ [illegible].

$= 0 \cdot 0 = 0.$

and so AHA not stuck anymore.

$$\textcircled{*} = \frac{1-0}{1+0} = 1 = \lim_{x\to\infty} \frac{x^2-1}{x^2+1}.$$

Confirming / Demonstrating / Proving what the geometric intuition suggested.

Def$^{\underline{n}}$ $\lim_{x\to\infty} f(x) = L$ (L finite or $L < \infty$) when there is some point on the x-axis after which $f(x)$ is arbitrarily close to L.

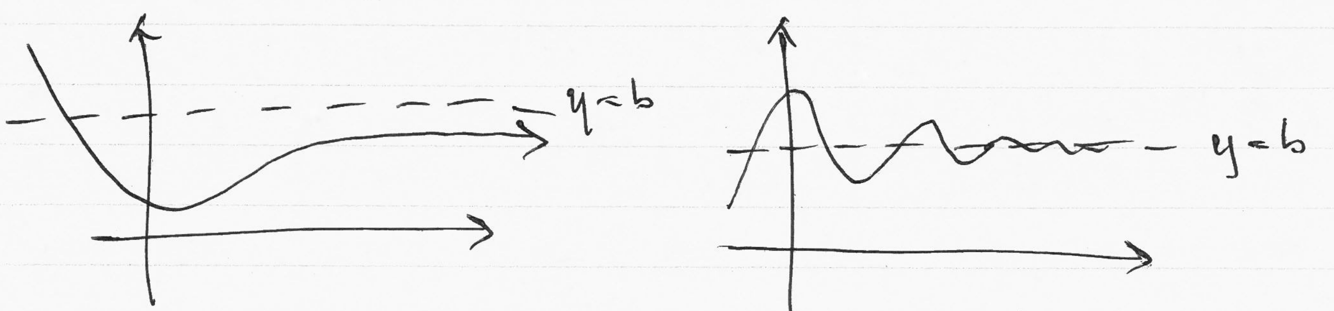
STRICTLY: $\forall \varepsilon > 0 \; \exists a \in \mathbb{R} : x > a \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$

Defn $\lim_{x \to -\infty} f(x) = L$ "..." before which "..."

$\forall \varepsilon > 0 \; \exists a \in \mathbb{R} : x < a \Rightarrow \underbrace{|f(x) - L| < \varepsilon}$

equivalent to $f(x) \in (L-\varepsilon, L+\varepsilon)$
(just for the sake of it)

EXAMPLE Functions can cross the HA: ~~on~~ For instance the following functions have HA, despite crossing them.



(ISOLATING DOMINANT TERM)

EXAMPLE, $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{\frac{1}{x^2}(3x^2 - x - 2)}{\frac{1}{x^2}(5x^2 + 4x + 1)}$

$= \lim_{x \to \infty} \frac{\left(\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}\right)}{\left(\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}\right)} = \lim_{x \to \infty} \frac{\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$

$= \frac{3 - 0 - 0}{5 + 0 + 0} = 3/5.$ Check w/ DESMOS.

<u>EXAMPLE</u> : by conjugation and isolating dom term.

$$\lim_{x\to\infty} \sqrt{x^2+1} - x \qquad \text{"} = \infty - \infty \text{"}$$

$$= \lim_{x\to\infty} \left[(x^2+1)^{\frac{1}{2}} - x\right]\left[\frac{(x^2+1)^{\frac{1}{2}} + x}{(x^2+1)^{\frac{1}{2}} + x}\right]$$

$$= \lim_{x\to\infty} \frac{(x^2+1) - x^2}{(x^2+1)^{\frac{1}{2}} + x}$$

$$= \lim_{x\to\infty} \frac{1}{(x^2+1)^{\frac{1}{2}} + x}$$

Dominant term is $\sqrt{x^2} = x$. so...

$$= \lim_{x\to\infty} \frac{\frac{1}{x} \cdot 1}{\frac{1}{x}\left[(x^2+1)^{\frac{1}{2}} + x\right]} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\left(\frac{x^2}{x^2} + \frac{1}{x^2}\right)^{\frac{1}{2}} + \frac{x}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} + 1} = \frac{0}{(1+0)^{\frac{1}{2}} + 1} = 0/2 = 0.$$

Confirm w/ DESMOS.

$$\sqrt{x^2} = |x|$$

<u>EXAMPLE</u>: $f(x) = \dfrac{(2x^2+1)^{\frac{1}{2}}}{3x-5}$ has three asymptotes.

$$\lim_{x\to\infty} \frac{(2x^2+1)^{\frac{1}{2}}}{3x-5} = \lim_{x\to\infty} \frac{\frac{1}{x}(2x^2+1)^{\frac{1}{2}}}{\frac{1}{x}(3x-5)}$$

because $x \to \infty$, $x > 0$ and $\sqrt{x^2} = |x| = x$

$$= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2}}\sqrt{2x^2+1}}{\frac{1}{\cancel{\sqrt{x^2}}\,x}(3x-5)} = \lim_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\left(\frac{3x}{x}-\frac{5}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{2+0}}{(3-0)} = \sqrt{2}/3.$$

Thus $x = \sqrt{2}/3$ is a horizontal asymptote.

$$\lim_{x\to-\infty} \frac{(2x^2+1)^{\frac{1}{2}}}{3x-5} = \lim_{x\to\infty} \frac{\frac{1}{x}(2x^2+1)^{\frac{1}{2}}}{\frac{1}{x}(3x-5)}$$

because $x \to -\infty$, $x < 0$ and $\sqrt{x^2} = |x| = -x$

$$= \lim_{x\to\infty} \frac{\frac{-1}{(x^2)^{\frac{1}{2}}}(2x^2+1)^{\frac{1}{2}}}{\frac{1}{x}(3x-5)} = \cdots = -\sqrt{2}/3.$$

So there is a HA at $x = -\sqrt{2}/3$.

The last, Vertical asymptote is at $x=5$ as

$$\lim_{x\to 5^+} f(x) = +\infty. \qquad \lim_{x\to 5^-} f(x) = -\infty.$$

§ Limits at Infinity

Note: $\infty + \infty = \infty$ but $\infty - \infty$ undefined.
$\infty \cdot \infty = \infty$ $\qquad \infty / \infty$ undefined.

NOTATION $\lim\limits_{x\to\infty} f(x) = \infty$ means $f(x)$ is "growing unbounded" as $x \to \infty$.
(The only alternative is having a HA).

STRICT $\lim\limits_{x\to\infty} f(x) = \infty \Leftrightarrow \forall y \exists a : x > a \Rightarrow f(x) > y$.

(i.e. you cannot pick y large enougen to "bound" $f(x)$).

There are similar defns for $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$

EXAMPLE $\lim_{x \to \infty} x^2 - x$

WRONG: $\lim_{x \to \infty} x^2 - x = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x = \underbrace{\infty - \infty = 0}_{\text{not true}}$

CORRECT: $\lim_{x \to \infty} x^2 - x = \lim_{x \to \infty} x(x-1) = \lim_{x \to \infty} x \cdot \lim_{x \to \infty} (x-1)$

$= \infty \cdot \infty = \infty.$

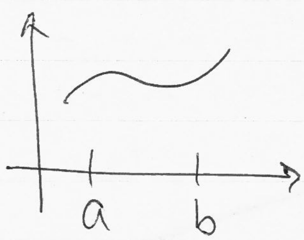
EXERCISE $\lim_{x \to \infty} \dfrac{3x^2 - x + 4}{2x^2 + 5x - 8}$

EXERCISE

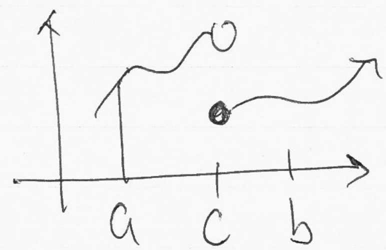
- $\lim_{x \to a} \left(\dfrac{x^2}{2} - \dfrac{1}{x} \right)$ for $a = 0^+$, 0^-, $2^{3/2}$, and -1.

§CONTINUITY

Basically: a function is continuous over some interval when you can draw it w/out lifting your pen.



"Continuous on (a,b)"



"discontinuous at c"

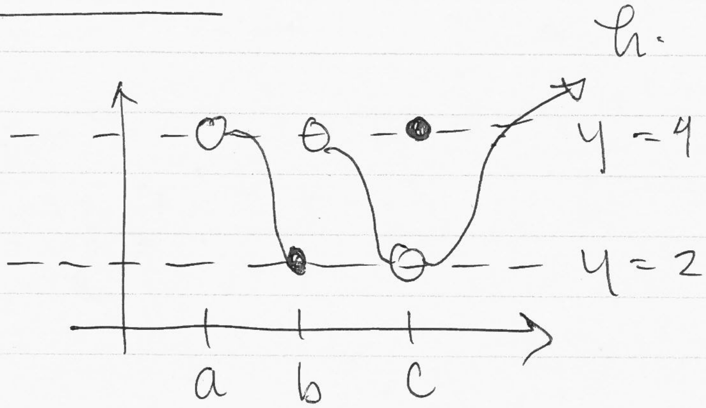
<u>Def^n</u> $f:\mathbb{R}\to\mathbb{R}$ a function is <u>continuous at a</u> when $\lim_{x\to a} f(x) = f(a)$.

Consequently, to ~~prove~~ show a function is cont. at a you must show three things:

(1.) $a \in \text{dom} f$, (2.) $\lim_{x\to a} f(x)$ exists, and

(3.) $\lim_{x\to a} f(x) = f(a)$.

EXAMPLE



h(x) is discont at $x = a, b, c$.

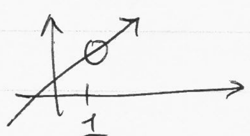
$x=a$ because $a \notin \text{dom} f \Rightarrow$ FAILS (1)

$x=b$ because $\lim\limits_{x\to b^-} h = 2 \neq 4 = \lim\limits_{x\to b^+} \Rightarrow$ FAILS (2)

$x=c$ because $\lim\limits_{x\to c} h = 2 \neq 4 = h(c) \Rightarrow$ FAILS (3)

EXAMPLE

$f(x) = \dfrac{x^2-x-2}{x-2}$ cannot be cont. at $x=2$ because $2 \notin \text{dom} f$.



Note, we can put the point back in. Because

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x+1)(x-2)}{(x-2)} = \lim_{x\to 2} x+1 = 3.$$

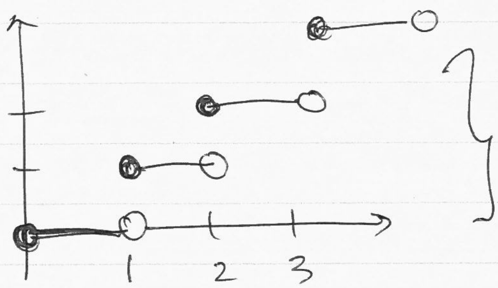
$$\& \quad g(x) = \begin{cases} 3 & x=2 \\ \dfrac{(x+1)(x-2)}{x-2} & \end{cases}$$

Restores continuity at this point — a ~~jump disc~~ removeable disc.

EXAMPLE $f(x) = \lfloor x \rfloor$ "floor" or "truncation"

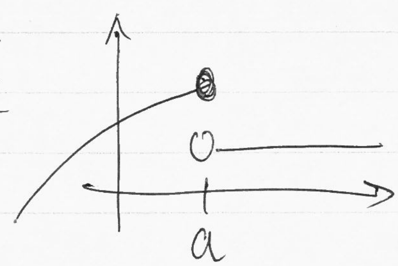
e.g. $\lfloor 2.3 \rfloor = 2$ $\quad \lfloor 2 \rfloor = 2$

Let $x \geq 0$ then



and so, notice $f(x)$ is disc at every integer.

EXAMPLE



f. Here we say f is left-continuous at $x=a$, but not right-continuous

Def$^{\underline{n}}$ Right Continuous / Left Continuous

A function f is continuous from the right at $x=a$ when $\lim_{x \to a^+} f(x) = f(a)$

and left-continuous when

$$\lim_{x \to a^-} f(x) = f(a).$$

QUESTION f cont at $a \Rightarrow f$ L-cont & R-cont at a?

YES!

Def$^{\underline{n}}$ f is cont on an open-interval $(a,b) \subseteq \mathbb{R}$ when $\forall x \in (a,b)$; $f(x)$ is cont.

Moreover, if $f(x)$ is R-cont at $x=a$ and L-cont at $x=b$ then $f(x)$ is continuous on $[a,b]$.

THM ~~EXAMPLES~~ When f and g are cont at a, $c \in \mathbb{R}$ then the following are also cont. at a

① $f+g$ ③ f/g provided $g(a) \neq 0$

② $f \cdot g$ ④ cf and $c \cdot g$.

Proof: Omitted.

Recall $\mathbb{R}[x] = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_k \in \mathbb{R}\}$ are polynomials and

$\{f/g : f, g \in \mathbb{R}[x] \text{ and } g \neq 0\}$

are rational functions.

Thm Polynomials are everywhere cont.

Rational functions are cont. on their domain.

EXAMPLE $\dfrac{x^3+2x-1}{5-3x}$ is continuous everywhere but $x = 5/3$.

Thm g cont at a and f cont at $g(a)$ then $(f \circ g)(x)$ is cont at a.

§ Intermediate Value Thm

Suppose f is cont on (a,b) and $f(a) < 0$ and $f(b) > 0$. Given that we cannot pick up our pens when drawing, $f(x)$ must pass through the $y=0$ axis

In other words $\exists\, c \in (a,b) : f(c) = 0$.

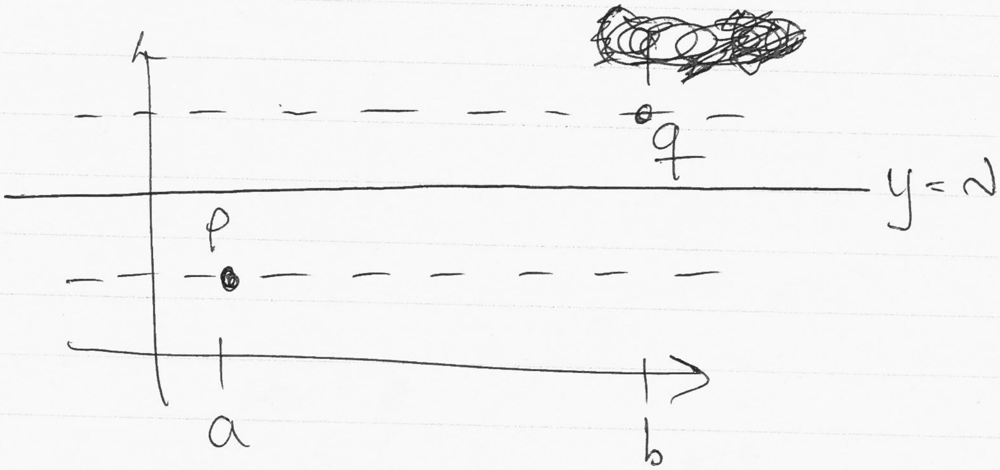
Thm IVT

Provided: • f is cont on $[a,b]$

• $N \in [f(a), f(b)]$

Then: $\exists\, c \in (a,b) : f(c) = N$.

In other words: You must cross $y=N$ to connect p and q below.



EXAMPLE: Show $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$ has a root in $(1,2)$.

Notice • $f(x)$ is a polynomial $\Rightarrow$ everywhere cont

• $f(1) = 4 - 6 + 3 - 2 = -1 < 0$

• $f(2) = 4\cdot 8 - 64 + 3\cdot 2 - 2 = 12 > 0$

Consequently,

We have $0 \in (f(1), f(2))$ and <u>by IVT</u>

$$\exists\, c \in (1,2) : f(c) = 0$$

Thus f has a root in $(1,2)$.