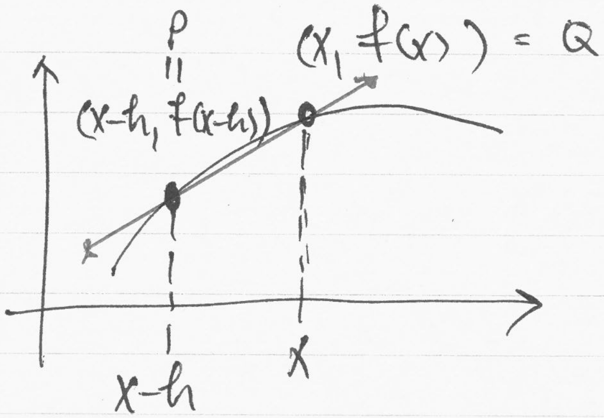
§LIMITS

Motivation Soon we will need to move one point into another to find the TANGENT:



want to move P close to but not equal to q.

(You need two points to form a line.)

Let us investigate

$$f(x) = \frac{\sqrt{x^2+100} - 10}{x^2}$$

near zero using a calculator.

Note $f(x) = f(-x)$ for our example.

x	$f(x)$	
1	0.049876	"left limit"
0.1	0.049999	
0.001	0.050000	calculator may report 0.
0.0001	0.050000	
↓	↓	
0	0.05	
↑	↑	
-0.0001	0.05000	"right limit"
-0.001	0.05000	
↑		

WARNING!

better.

We say $\lim_{x \to 0} \frac{\sqrt{x^2+100} - 10}{x^2} = 0.05 = 5/100$

<u>Def^n</u> Limit (loose)

We say $\lim_{x \to a} f(x) = L$ when $f(x)$ becomes ~~arbitrary~~ <u>arbitrarily</u> close to L (the limit) when x is <u>sufficiently</u> close to a (on both sides) but <u>not</u> a.

<u>Def^n</u> Limit (strict) *You dont need to know this *

$\lim_{x \to a} f(x) = L$ when $\forall \varepsilon > 0; \exists \delta > 0 :$

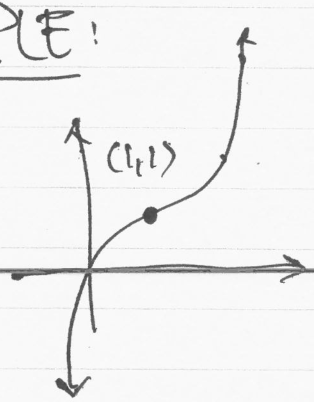
δ = "delta"
ε = "epsilon"

$x \in (a-\delta, a) \cup (a, a+\delta) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$

English: the limit as x approaches a ~~along~~ of $f(x)$ is L when for any arbitrarily small positive ε there is another ~~arbit~~ small enough δ such that $f(x)$ is ε-away from L.

Limits are usually easy to calculate when you have the <u>graph</u>.

<u>EXAMPLE</u>:



Let $f(x) = (x-1)^3 + 1$

From graph:

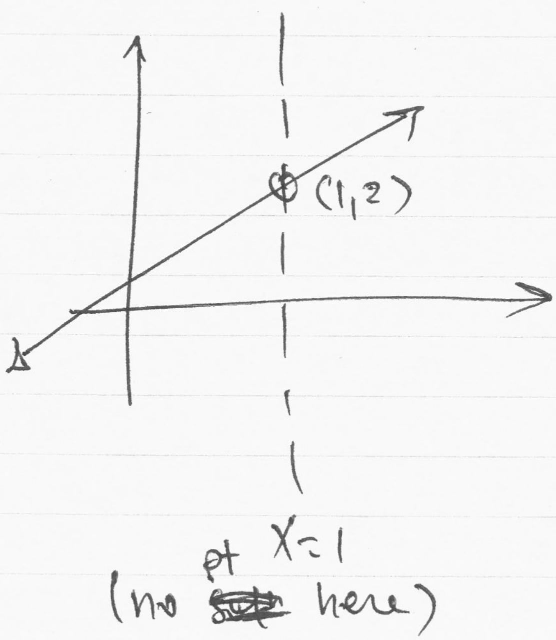
$$\lim_{x \to 1} (x-1)^3 + 1 = 1$$

<u>NOTATION</u> $x \to a$ means x gets arbitrarily close to, but not equal to, a.

<u>EXAMPLE</u>: $f(x) = \dfrac{x^2-1}{x-1}$. Find $\lim\limits_{x \to 1} f(x)$.

Notice $1 \notin \operatorname{dom} f$.

Recall "o" means point is <u>removed</u>.

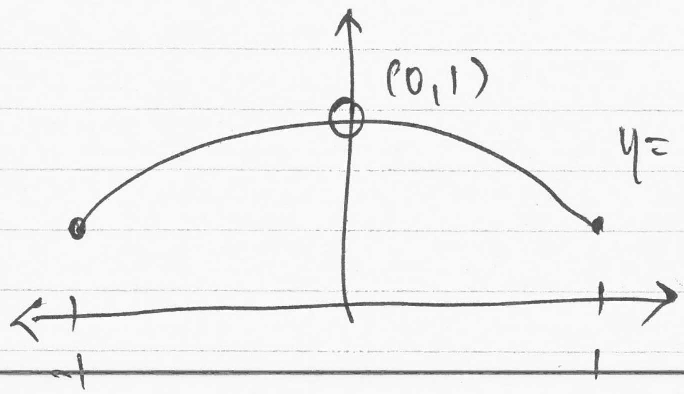


From graph:

$$\lim_{x \to 1} f(x) = 1.$$

Again, x never becomes 1.

EXAMPLE $\lim\limits_{x \to 0} \frac{\sin x}{x} = ?$



$y = \frac{\sin x}{x}$

From the graph:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

We do not yet have the capacity to demonstrate these limits algebraically.

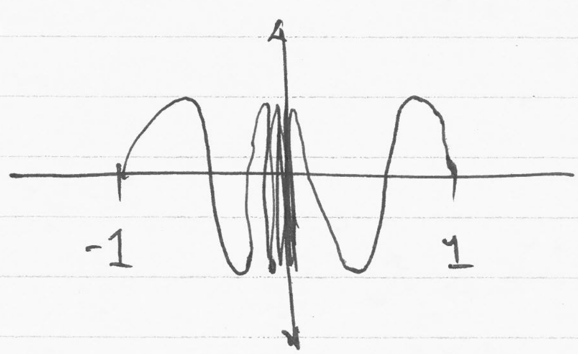
EXAMPLE Let $f(x) = \sin \frac{\pi}{x}$

Numerically $f\left(\frac{1}{10}\right) = \sin 10\pi = 0$

$f\left(\frac{1}{100}\right) = \sin 100\pi = 0$

$f\left(\frac{1}{1000}\right) = \sin 1000\pi = 0 \ \ldots$

GUESS: $\lim\limits_{x \to 0} \sin \frac{\pi}{x} = 0.$ CHECK w/ plot

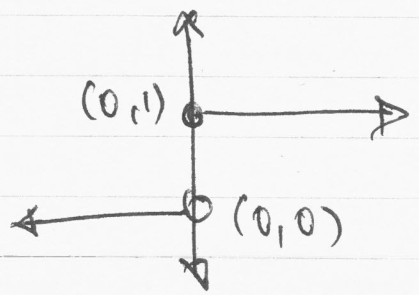


There are infinitely many instances of the sin curve about 0.

§One-Sided Limits

EXAMPLE (Heaviside Function)

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{otherwise} \end{cases}$$



$\lim_{x \to 0} H(t)$ DOES NOT EXIST.
= DNE.

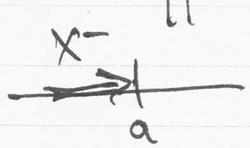
But the left and right limits exist and

$$\lim_{t \to 0^-} H(t) = 0 \qquad \lim_{t \to 0^+} H(t) = 1.$$

Def^n Left Hand Limit (Good Enough)

$$\lim_{x \to a^-} f(x) = L$$

when x approaches a from the left on the real line:



$f(x)$ approaches L.

Defn Right Hand Limit (Strict)

$$\lim_{x \to a^+} f(x) = L$$

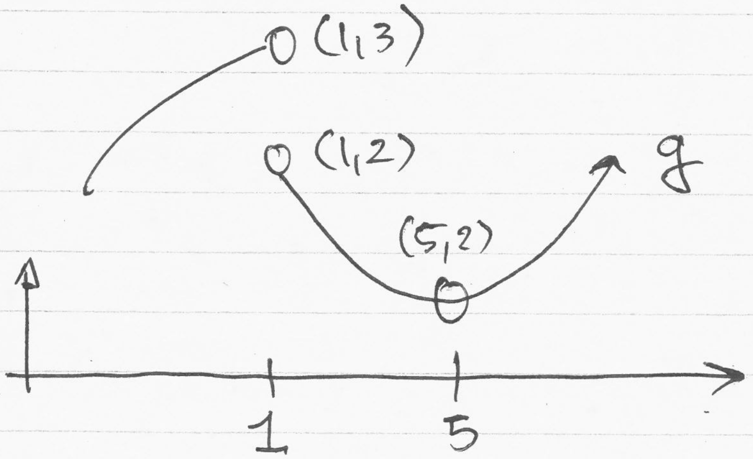
when $\forall \varepsilon > 0; \exists \delta > 0 : x \in (a, a+\delta) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$

--- again, you will not be tested on strict defn.

$$\boxed{\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)}$$

i.e. the limit "exists" only when the left & right limits exist and are equal.

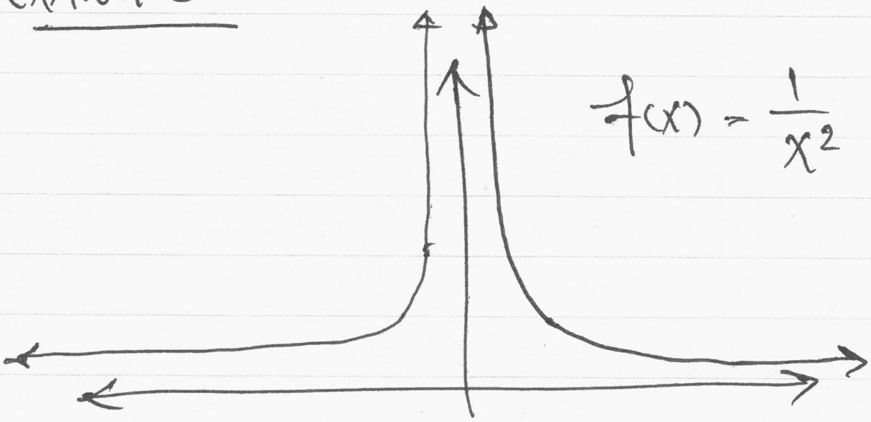
QUESTION: Read the following off the graph.



- $g(1) = 3$, • $g(5) =$ DNE
- $\lim_{x \to 1^+} g = 2$, • $\lim_{x \to 5} g = 2$
- $\lim_{x \to 1^-} g = 3$
- $\lim_{x \to 1} g$ DNE

§ Infinite Limits

EXAMPLE

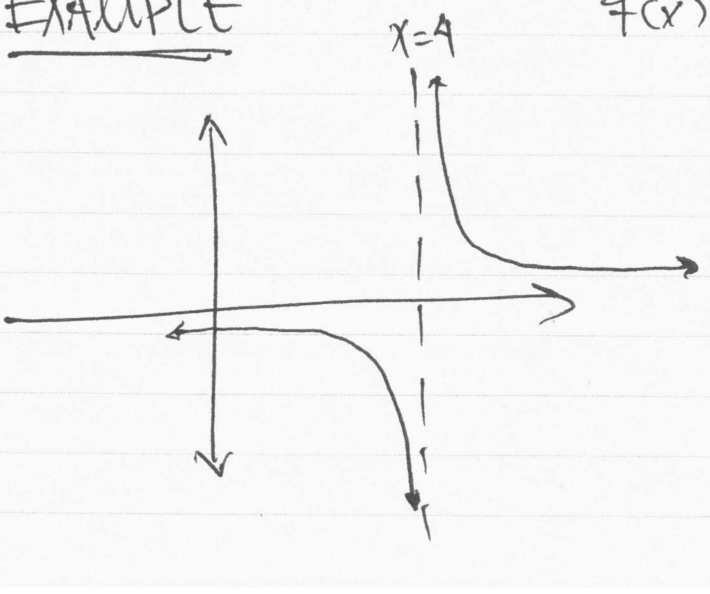


$$f(x) = \frac{1}{x^2}$$

Here $\lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$ — approaching infinity is fundamentally different than "equalling" infinity.

∞'s always come w/ limits — this is in part why we need them.

EXAMPLE



$$f(x) = \frac{1}{x-5}$$

$$\lim_{x \to 0} \frac{1}{x-5} \quad \text{DNE}$$

$$\lim_{x \to 0^+} \frac{1}{x-5} = +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x-5} = -\infty.$$

Def$^{\underline{n}}$ Vertical Asymptote

When $\lim_{x \to a^+} f(x) = +\infty, -\infty$

$\lim_{x \to a^-} f(x) = +\infty, -\infty$

then $x = a$ is called a vertical asymptote of $f(x)$.

EXERCISE

ⓘ $\lim_{x \to 1} \frac{2-x}{(x-1)^2}$ ⓘⓘ $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$

§LIMIT LAWS

Now we tie limits back to algebra as you dont have a plotter on tests!

Limit Laws

Suppose $c \in \mathbb{R}$ and $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ exist.

Then:

1. $$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

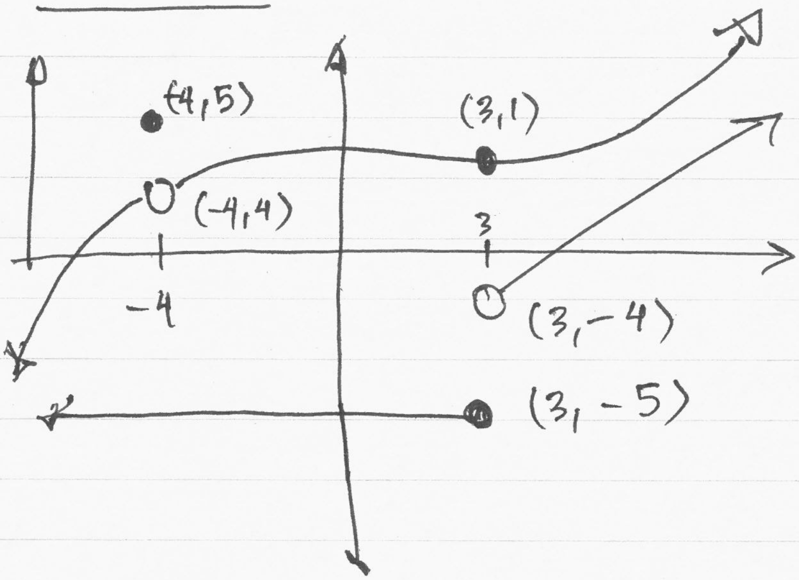
2. $$\lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x)$$

3. $$\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

4. $$\lim_{x \to a} f(x)/g(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

provided $\lim_{x \to a} g(x) \neq 0$.

EXAMPLE



$$\lim_{x\to -4} f\cdot g = \lim_{x\to -4} f \cdot \lim_{x\to -4} g = 4\cdot(-5) = -20$$

$$\lim_{x\to 3} f+g = \lim_{x\to 3} f + \lim_{x\to 3} g = 1 + \text{DNE} = \text{DNE}$$

Prop^n Direct Sub. Prop.

If f is a polynomial:

$$f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \; : \; a_j \in \mathbb{R}$$

or rational function (allow division and n^{th} roots).

Provided $a \in \operatorname{dom} f$ then

$$\lim_{x \to a} f(x) = f(a).$$

EXAMPLE Now we can do:

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x \to 1} (x+1) = 2.$$

EXAMPLE

$$\lim_{t \to 0} \frac{(t^2+9)^{\frac{1}{2}} - 3}{t^2}$$

$$= \lim_{t \to 0} \frac{(t^2+9) - 9}{t^2\left[(t^2+9)^{\frac{1}{2}} + 3\right]}$$

$$= \lim_{t \to 0} \frac{1}{(t^2+9)^{\frac{1}{2}} + 3}$$

$$= \frac{1}{6}.$$

EXAMPLE

$$\lim_{x\to 0}|x| = \begin{cases} \lim_{x\to 0} x & x \geq 0 \\ \lim_{x\to 0} -x & x < 0 \end{cases}$$

$$= \begin{cases} \lim_{x\to 0^+} x & x \geq 0 \\ \lim_{x\to 0^-} -x & x < 0 \end{cases} = \begin{cases} 0 & x \geq 0 \\ 0 \end{cases} = 0$$

§ Squeeze Theorem

Recall $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ — we need algebra now.

Thm Squeeze Thm

Provided:
- $f(x) \leq g(x) \leq h(x)$ for x near a, $x \neq a$.
- $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

then $\lim_{x\to a} g(x) = L$.

WORKSHEET.

EXAMPLE $\lim\limits_{x\to 0} x^2\cdot \sin\left(\frac{1}{x}\right) = 0$

Notice $-1 \le \sin\left(\frac{1}{x}\right) \le 1 \qquad x \neq 0$

$\Rightarrow \quad -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2 \qquad$ (x^2 is positive)

By squeeze thm...

$$\Rightarrow \lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2 \sin\frac{1}{x} \le \lim_{x\to 0} x^2$$

$$\Rightarrow \quad 0 \le \lim_{x\to 0} x^2 \sin\frac{1}{x} \le 0$$

$$\Rightarrow \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$ by squeeze thm.

EXERCISE

- $\lim\limits_{x\to 3} \left(2x + |x-3|\right)$
- $\lim\limits_{h\to 0} \dfrac{(2+h)^3 - 8}{h}$.