Mappings and Functions

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Definition 1 (Mapping)

A mapping (loosely speaking) is a relating of elements from one set to another. We write

$$f : A \to B$$
$$a \mapsto b$$

to denote $a \in A$ maps to $b \in B$.

It is typical to write $f(a) = b$ when $a \mapsto b$.

Question 1

The mapping f given by a table of values.

A	B
0	0
-1	1
1	1
-2	4
2	4

What is $f(1)$? What is $f(9)$?

Answer

$f(1) = 1$. $f(9) =$ undefined.

Definition 2 (Domain)

Let $f : A \to B$. The domain of f is the subset of A for which the function is defined:

$$\operatorname{dom} f \coloneqq \{a \in A : f(a) \text{ is defined}\}$$

In practice finding the domain of a function usually boils down to ensuring we never divide by zero or take square-roots of negative numbers.

Question 2

What is dom f for the mapping f given by the following table?

A	B
0	0
-1	1
1	1
-2	4
2	4

Answer

$$\operatorname{dom} f = \{0,\ -1,\ 1,\ -2,\ 2\}\,.$$

Definition 3 (Range)

Let $f : A \to B$. The range of f is simply

$$\operatorname{rng} f \coloneqq \{f(a) : a \in \operatorname{dom} f\} \subseteq B.$$

Intuitively, these are the points which are "reachable" by f.

Question 3

What is $\operatorname{rng} f$ for the mapping f given by the following table?

A	B
0	0
-1	1
1	1
-2	4
2	4

Answer

$$\begin{aligned}\operatorname{rng} f &= \{f(a) : a \in \operatorname{dom} f\} \\ &= \{f(a) : a \in \{0, -1, 1, -2, 2\}\} \\ &= \{f(0), f(-1), f(1), f(-2), f(2)\} \\ &= \{0, 1, 1, 4, 4\} \\ &= \{0, 1, 4\}\end{aligned}$$

Definition 4 (Graph)

Let $f : \mathbb{R} \to \mathbb{R}$. The graph of $x \mapsto y$ is the collection of points given by

$$\mathcal{G}(f) \coloneqq \{(x, y) : x \in \operatorname{dom} f \textbf{ and } x \mapsto y\} \subseteq \mathbb{R} \times \mathbb{R}$$

equivalently

$$(x, y) \in \mathcal{G}(f) \iff y = f(x).$$

Example 5

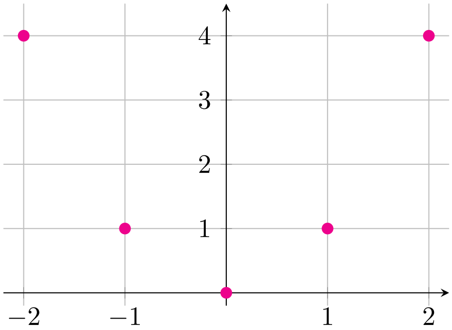
For our mapping f we have

$$\mathcal{G}(f) = \{(0, 0), (1, 1), (-1, 1), (2, 4), (-2, 4)\}$$

which is an encoding of the points into a set.

Example 6

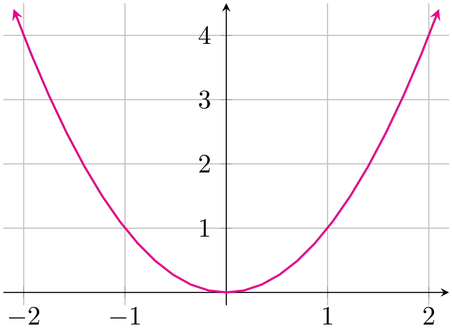
Alternatively we can encode the points of the graph using an illustration:



Note: A graph is any encoding of the points from $\mathcal{G}(f)$.

Question 4

The following is a graph of the mapping $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto x^2$.



What is the graph in set form?

Answer

$\mathcal{G}(f) = \{(x, x^2) : x \in \mathbb{R}\}.$

Question 5

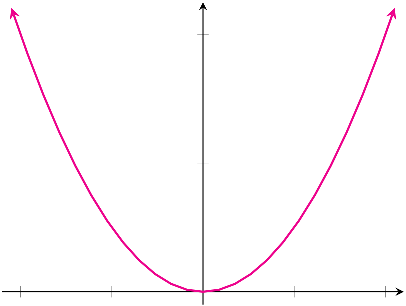
Which graph satisfies

$$f(a) = b_0 \textbf{ and } f(a) = b_1 \implies b_0 = b_1$$

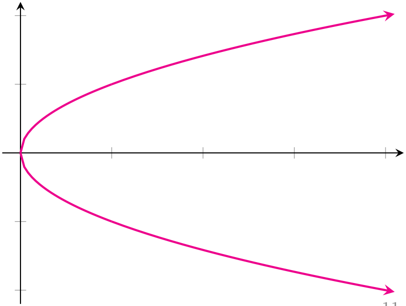
or equivalently

$$(a, b_0) \in \mathcal{G}(f) \textbf{ and } (a, b_1) \in \mathcal{G}(f) \implies b_0 = b_1.$$

f_0



f_1



Definition 7 (Vertical Line Test)

A mapping

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto y$$

satisfies the vertical line test when

$$f(x) = y_0 \textbf{ and } f(x) = y_1 \implies y_0 = y_1$$

For this course we are mostly interested in real valued functions.

Definition 8 (Real Valued Function)

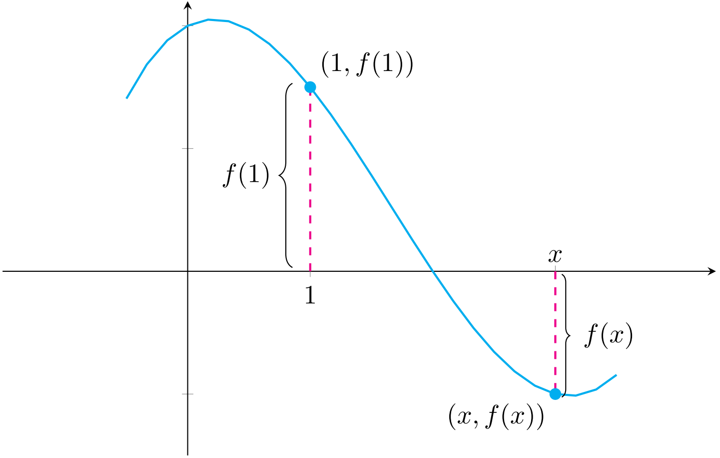
When f is a mapping given by

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto y \end{aligned}$$

and f satisfies the vertical line test then f is also a function.

Here x is called the independent variable and y the dependent variable because the value of y depends on x — this is why we write $y = f(x)$.

Example 9



Question 6

Let two mappings be given by

i. $x + y = 4$ ii. $x^2 + y^2 = 4.$

Which correspond to functions?

(Use desmos)

Answer

Only i.

Question 7

Show the mapping given by

$$x^2 + y^2 - 4$$

does not satisfy the vertical line test.

Answer

We have $f(x) = \pm\sqrt{4 - x^2}$ so

$$(1,\ \sqrt{3}) \in \mathcal{G}(f) \textbf{ and } (1,\ -\sqrt{3}) \in \mathcal{G}(f) \textbf{ and } \sqrt{3} \neq -\sqrt{3}$$

and thus the vertical line test fails.

Question 8

Write an algebraic rule for the horizontal line test and show

$$f(x) = (x - 1)(x - 2)$$

does not satisfy it.

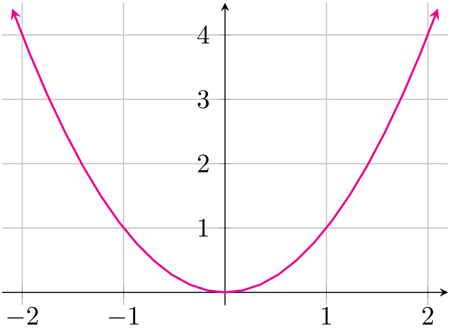
Answer

$f(a) = y$ **and** $f(b) = y \implies a = b$.

$f(1) = 0$ **and** $f(2) = 0$ **and** $1 \neq 2$.

Question 9

What is the “natural” domain and range of the function $f(x) = \dfrac{1}{x^2 - 1}$?



Answer

$\operatorname{dom} f = \mathbb{R} - \{-1, 1\}$. $\operatorname{rng} f = \mathbb{R}$.

Question 10

What is the natural domain and range of $y = 3 + \sqrt{x}$?

Answer

$\operatorname{dom} f = [0, \infty)$ and $\operatorname{rng} f = [3, \infty)$

Exercise

Find the domain and range of the following function and determine whether it is even, odd, or neither:

$$f(x) = \frac{1}{1 - x^2} - \frac{1}{1 - x}.$$

Plot it with desmos to confirm your answer.

Definition 10 (Increasing Function)

Let $f : \mathbb{R} \to \mathbb{R}$ and let $I \subseteq \operatorname{dom} f$. f is increasing on I when

$$x_0, x_1 \in I \textbf{ and } x_0 < x_1 \implies f(x_0) < f(x_1)$$

and decreasing on I when

$$x_0, x_1 \in I \textbf{ and } x_0 < x_1 \implies f(x_0) < f(x_1).$$

Question 11

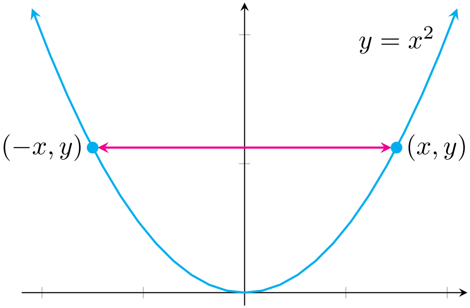
According to this definition. Is the function $f(x) = 5$ increasing or decreasing?

Answer

Neither! Notice the use of strict inequalities.

Question 12

The following parabola $f(x) = x^2$ is a even function because of symmetry about the $y = 0$ axis. Write the algebraic condition for being even.



Answer

The following are equivalent and correct

1. $(x, y) \in \mathcal{G}(f) \iff (-x, y) \in \mathcal{G}(f)$,

Definition 11 (Even Function)

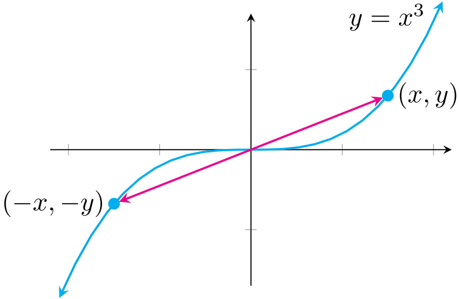
Let $f : \mathbb{R} \to \mathbb{R}$ be a real-function. Then f is even when

$$f(x) = f(-x)$$

for every $x \in \operatorname{dom} f$.

Question 13

The following cubic $f(x) = 3^2$ is an odd function because of symmetry about the $y = x$ axis. Write the algebraic condition for being odd.



Answer

The following are equivalent and correct

1. $(x,\, y) \in \mathcal{G}(f) \iff (-x,\, -y) \in \mathcal{G}(f)$
2. $f(x) = -f(-x)$ for all $x \in \operatorname{dom} f$

Definition 12 (Odd Function)

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-function. Then f is odd when

$$f(x) = -f(-x)$$

for every $x \in \operatorname{dom} f$.

Question 14

Is there an everywhere decreasing even function?

Answer

No! Because every even function f has $f(a) = f(-a)$ with $a > 0 > -a$.

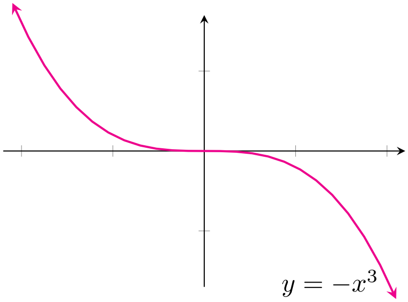
This violates the condition of decreasing.

Question 15

Is there an everywhere decreasing odd function?

Answer

Yes! $f(x) = -x^3$ is odd and everywhere decreasing.



Operations on Functions

Like numbers, functions can be added/subtracted and multiplied/divided.

Proposition 1

Given two real functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ then

1. $f(x) + g(x) = (f + g)(x)$, and
2. $f(x) \cdot g(x) = (f \cdot g)(x)$.

Example 13 (Adding two functions)

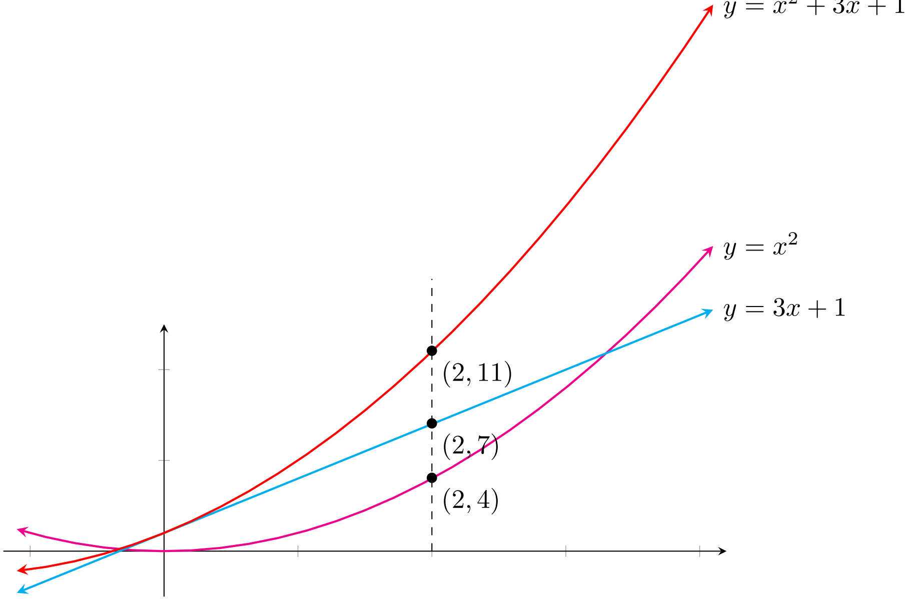
Let $f(x) = x^2$ and $g(x) = 3x + 1$.

Notice

$$f(2) + g(2) = (2^2) + (3 \cdot 2 + 1) = 11$$

and

$$(f + g)(2) = (x^2 + 3x + 1) = (2^2 + 3 \cdot 2 + 1) = 11.$$


$y = x^2 + 3x + 1$
$y = x^2$
$y = 3x + 1$
$(2, 11)$
$(2, 7)$
$(2, 4)$

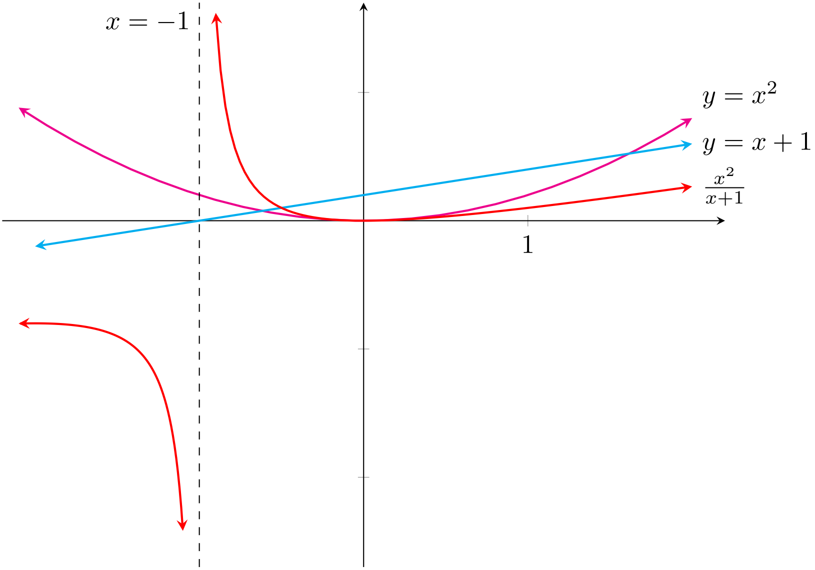
Example 14 (Dividing Two Functions)

Let $f(x) = x^2$ and $g(x) = x + 1$.

Plot

$$\frac{f}{g} = \frac{x^2}{x+1}$$

and compare dom and rng for f, g, and $\frac{f}{g}$.


$x = -1$
$y = x^2$
$y = x + 1$
$\frac{x^2}{x+1}$
1

Definition 15 (Function Composition)

Suppose $A, B, D \subseteq \mathbb{R}$ and

$$f : B \to C \qquad\qquad g : A \to B$$
$$b \mapsto c \qquad\qquad a \mapsto b$$

The function composition of f and g is defined to be

$$(f \circ g)(x) : A \to B \to C$$
$$a \mapsto b \mapsto c$$

or in the function form

$$(f \circ g)(x) = f(g(x)).$$

Question 16

Does $f \circ g = g \circ f$?

Answer

No! Notice when $f(x) = x^2$ and $g(x) = x - 3$ that

$$(f \circ g)(x) = f(g(x)) = f(x-3) = (x-3)^2 = x^2 - 6x + 9$$

but

$$(g \circ f)(x) = g(f(x)) = g(x^2) = (x^2) - 3 = x^2 - 3.$$

Exercise

Are there two functions f and g such that

1. $f \circ g = g \circ f$?
2. $f \circ g = 0$?
3. $f \circ g = 1$?

Transforming Functions

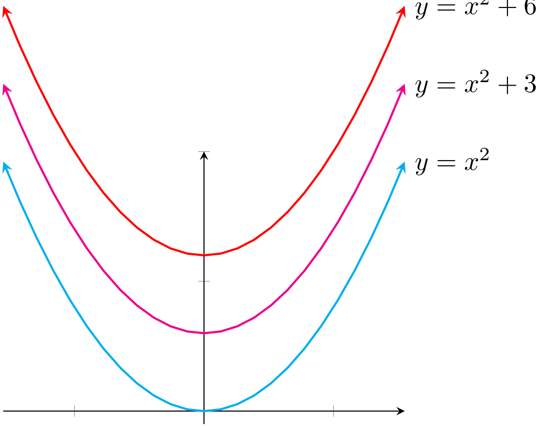
Shifting

Let $f : \mathbb{R} \to \mathbb{R}$.

Vertical Shift $y = f(x) + k$ shifts the graph up when $k > 0$ and down otherwise.

Horizontal Shift $y = f(x + h)$ shifts the graph left when $h > 0$ and right.

Vertical Shift



Question 17

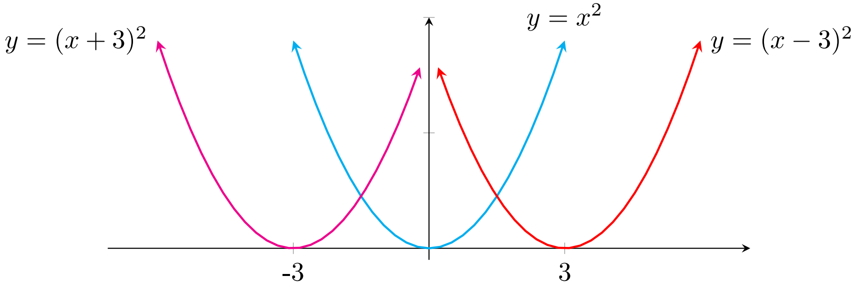
Let $f : \mathbb{R} \to \mathbb{R}$. What function would you compose with f to shift it left $a \in R$ units?

Answer

Composing $g(x) = x - a$ with a f gives

$$f \circ g = f(g(x)) = f(x - a).$$

Horizontal Shift



Transforming Functions

Let $f : \mathbb{R} \to \mathbb{R}$.

Scaling $y = f(cx)$ stretches horizontally when $|c| > 1$ and contracts horizontally otherwise.

$y = cf(x)$ stretches vertically when $|c| > 1$ and contracts vertically otherwise.

Reflection $y = -f(x)$ reflects across $x - axis$.

$y = f(-x)$ reflects across $y - axis$.

Exercise

Use desmos to investigate scaling and reflecting on the function $f(x) = \sqrt{x}$.

Next Class

1. Special functions.
2. Trigonometry.