Real Numbers and the Real Line

MATH 134

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Question

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Answer

Imprecisely: the set of (possible infinitely long) “decimal numbers”.

More precise: The set of limits of rational sequences. (You do not need to understand this – it is just a fun fact.)

Closed Ray

$$[a, \infty) = \{x : x \geq a\} \subseteq \mathbb{R}$$

Open Ray

$$(-\infty,\ b) = \{x : x < b\} \subseteq \mathbb{R}$$

Number line

$$\mathbb{R} = (-\infty,\ \infty).$$

Open Interval

$$(a,\, b) = \{x : a < x < b\} = \{x : x > a \text{ and } x < b\}$$

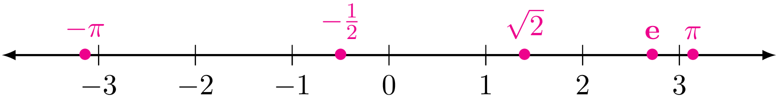
Closed Interval

$$[a,\, b] = \{x : a \le x \le b\} = \{x : x \ge a \text{ and } x \le b\}$$

Half-Open / "Clopen" Interval

$$[a,\, b) = \{x : a \le x < b\} = \{x : x \ge a \text{ and } x < b\}$$

There are "no holes" in the number line. (This is a statement about the density of the real numbers).



Sets of real numbers

Sets of real numbers are written using curly braces

$$A = \{1, 2, 3, \ldots\} \qquad B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\} \qquad C = \{x : x^2 > 0\}$$

Definition (Real Number)

A real number is any number on the number line. It may be an integer, a positive or negative number, a fraction or a decimal.

Example

Real numbers that are rational (i.e. expressible as fractions)

$$2, \qquad 0, \qquad -7, \qquad \frac{2}{3}, \qquad -\frac{4}{5} = -0.8\dot{0}, \qquad \frac{1}{2} = 0.333\ldots.$$

Real numbers that are not rational (i.e. irrational)

$$\sqrt{2} = 1.4142\ldots \qquad \pi = 3.141\ldots \qquad \mathbf{e} = 2.71\ldots$$

Question

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Answer

Yes! We can extend the real numbers to the complex numbers and quaternions (neither are covered in this course).

Example

The complex number $\mathbf{i} = \sqrt{-1}$ is not real (i.e. it is imaginary).

Definition (Real Number)

For any $x \in \mathbb{R}$ we define the absolute value by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We may think of $|x|$ as the distance of x from 0.

The absolute value satisfies (these and others found on page AP5).

$$|a| \geq 0 \qquad |a \cdot b| = |a| \cdot |b| \qquad \sqrt{x^2} = |x|$$

Set Notation

Let A and B be sets (i.e. unordered collections of numbers).

Intersection $A \cap B = \{x : x \in A \text{ and } x \in B\}$,

Union $A \cup B = \{x : x \in A \text{ or } x \in B\}$,

Element $x \in A$ means x is an element of,

Not Element $x \notin A$ means x is not an element of A.

Example

Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$.

$$A \cap B = \{\} = \varnothing \qquad A \cup B = \{1, 2, 3, 4, 5, 6\} \qquad 1 \in A \qquad 1 \notin A.$$

Standard Sets

Naturals $\mathbb{N} = \{1, 2, 3, \ldots\}$,

Integers $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$,

Rationals $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$,

Reals $\mathbb{R} = \{x : x \text{ is a real number}\}$.

In Class Activity

Exercise

Solve (i.e. find all possible x satisfying) the following

1. $\dfrac{4}{x+1} = -\dfrac{3}{x}$,
2. $2x^4 - 15x^2 - 8 = 0$,
3. $\dfrac{x^2}{3} \geq 1 - x^2$, and
4. $x^2 < x$.

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Answer

1. $x = -\frac{3}{7}$ 2. $x = \pm 2\sqrt{2}$ 3. $x \notin \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ 4. $x \in (0, 1)$.

Exercise

Solve the following systems of equations. That is, find all x which satisfies the equation simultaneously and express your answer using interval notation.

1. $\left\{ \dfrac{x^2}{3} \geq 1 - x^2,\ x^2 < x \right\}$
2. $x^3 > 9x$.

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Answer

1. $x \in \left[\sqrt{\frac{3}{2}}, 1 \right)$

2. $x \in (-3, 0) \cup (3, \infty)$

Exercise

Solve the following and use interval notation when possible to express your solution.

1. $|x - 3| = 5$,
2. $3|x| - 7 \geq 11$,
3. $|2x - 7| < 4$.

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3. $|2x-7|<4$.

Answer

1. $x \in \{-2, 8\}$, 2. $x \notin (-6, 6)$, 3. $x \in \left(\frac{3}{2}, \frac{11}{2}\right)$.

We are going to reflect on how to solve the following problem...

Question

Find a quadratic polynomial $p(x)$ such that:

1. $p(x) > 0$ when $x < 1$ and $x > 3$,
2. $p(x) = 0$ if $x = 1$ or $x = 3$, and
3. $p(x) < 0$ when $x \in (1, 3)$.

Exercise

First we are going to understand the statement of the question.

For each of these concepts, write down what it means or write **stuck**.

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3. "$p(x)$ is zero if $x = 1$ or $x = 3$". (Can you express this in two equations?)

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1. A quadratic polynomial.
2. A parabola.
3. "$p(x)$ is zero if $x = 1$ or $x = 3$". (Can you express this in two equations?)
4. The polynomial $p(x)$ is of your choosing.

Exercise

From the information we have already, we want to draw a picture.

This will give us a geometric understanding of the problem.

For each of these tasks, explain what to do, or write **stuck**.

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2. Will the parabola $p(x)$ open upwards or downwards? (How do you know?)

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1. What does “$p(x)$ is zero if $x = 1$ or $x = 3$” look like in a picture?
2. Will the parabola $p(x)$ open upwards or downwards? (How do you know?)
3. Approximately where will the vertex or the parabola $p(x)$ be?

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4. Is there only one possible value for the vertex, or are there many possible values for the vertex?

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Were **stuck** on any of these? Why?

Exercise

There are three common ways to express a parabola $p(x)$:

1. **Standard form**: $p(x) = ax^2 + bx + c$,
2. **Vertex form**: $p(x) = a(x-h)^2 + k$,
3. **Root form**: $p(x) = a(x-r_1)(x-r_2)$.

Reflect: What are the different forms useful for? Do the coefficients represent anything, or do they give us any information about the behaviour of the parabola? Which of these forms do you feel might be the most useful for our problem? Why?

Check: Is the "a" in the Standard form the same as the "a" in the vertex form and the "a" in the root form?

Check: What happens to the parabola if "a" is a positive number?

Exercise

Using one of the forms, make a parabola $p(x)$ that solves our original question.

Check that it has the properties that you claim it has. If it does not and you get **stuck**, then write down which property your polynomial is missing, and how you can fix it. Write **AHA**! when you get it.

Exercise (Reflect)

What would happen if in the original question the important numbers were not $x = 1$ and $x = 3$, but instead they were $x = -1$ and $x = 4$. Can you solve this new problem by adapting your solution to the original problem?

Generalize

What else can you change in the original problem? Find two different ways to modify the original problem statement.

Reflect

Can you solve these new problems by adapting your solution to the original problem? (It's okay if the answer is no.) Do you prefer coming up with new problems (the *Theory Builder*), or do you prefer , or do you prefer ?

Reflect

Which of these describe your style of mathematics? What types of people work well together? I like . . .

1. coming up with new generalizations and new problems that no one has ever thought of before. I am a Theory Builder.
2. solving problems no one has ever solved before. I am a Problem Solver.
3. finding the perfect problem for each problem solver. I am a Connector.
4. explaining mathematics in new ways. I am a Communicator.

For more information, read the essay "The Two Cultures of Mathematics" by Timothy Gowers.

Question

There is an $p(x)$, of your choosing, such that:

1. $p(x)$ is positive if x is not 0 or 1, and
2. $p(x)$ is zero if $x = 0$ or $x = 1$.

Exercise

Solve the above problem using our problem solving framework (**stuck**, **AHA**!, **check**, **refelct**, **specialize**, and **generalize**.