Total and Partial Orderings

Introduction to Computer Programming

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Rationals Continued

Question

When should two rationals be equal? Use your definition to implement `__eq__(self, other)` to `Rational`.

Magic Methods

Comparison	Magic Method
`<`	`__gt__`
`<=`	`__le__`
`==`	`__eq__`
`!=`	`__ne__`
`>`	`__gt__`
`>=`	`__ge__`

Question

Suppose you have a set (of objects) and want to sort them.

What properties (precisely) does the comparison operator $\prec$ (read: "precedes or is") need to satisfy for such an ordering to be possible and unique.

Question

Add the total ordering to the `Rational` class and then sort a list of Rational objects.

Not testable material follows.

Definition (Partial Order)

A relation $\prec$ on a collection of objects A defines a partial order when

Reflexive $\forall a \in A$

$$a \prec a$$

Antisymmetric $\forall a, b \in A$

$$a \prec b \texttt{ and } b \prec a \implies a = b$$

Transitive $\forall a, b, c \in A$

$$a \prec b \texttt{ and } b \prec c \implies a \prec c$$

Definition (Total Order)

A partial ordering $\prec$ is a total ordering if it has the the additional property that

$$\forall a, b \in A;\ a \prec b \ \texttt{or} \ b \leqslant a.$$

Question

Show that $\leqslant_{\mathbb{Q}}$ (the ordering on rationals) given by

$$\frac{a}{b} \leqslant_{\mathbb{Q}} \frac{c}{d} \overset{\text{def}}{\iff} ad \leqslant_{\mathbb{Z}} cb$$

(where $\leqslant_{\mathbb{Z}}$ is the usual total ordering over the integers) is a total ordering.

Answer (Antisymmetry)

For any $\frac{a}{b} \in \mathbb{Q}$

$$a \leqslant_{\mathbb{Z}} a \implies ab \leqslant_{\mathbb{Z}} ab \implies \frac{a}{b} \leqslant_{\mathbb{Q}} \frac{a}{b}$$

and thus $\geq$ is antisymmetric. $\square$

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Answer (Reflexive)

For any a, $b \in \mathbb{Z}$ we have

$$ab \leqslant_{\mathbb{Z}} ab \implies \frac{a}{b} \leqslant_{\mathbb{Q}} \frac{a}{b}$$

by definition. $\square$

Answer (Transitivity)

Assume a, b, c, d, e, f are positive integers then: (note we let $\leqslant = \leqslant_{\mathbb{Z}}$ for readability)

$$\frac{a}{b} \leqslant_{\mathbb{Q}} \frac{c}{d} \quad \texttt{and} \quad \frac{c}{d} \leqslant_{\mathbb{Q}} \frac{e}{f}$$

This argument can be repeated for the other cases where negatives obligate us to flip the sign.

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Assume a, b, c, d, e, f are positive integers then: (note we let $\leqslant = \leqslant_{\mathbb{Z}}$ for readability)

$$\frac{a}{b} \leqslant_{\mathbb{Q}} \frac{c}{d} \text{ and } \frac{c}{d} \leqslant_{\mathbb{Q}} \frac{e}{f}$$

$$\implies ad \leqslant bc \text{ and } cf \leqslant de \qquad \text{by def}$$

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&\implies adf \leqslant bde && \text{trans of } \leqslant \\
&\implies af \leqslant be && \div d \\
&\implies \frac{a}{b} \leqslant_{\mathbb{Q}} \frac{e}{f} && \text{by def. } \square
\end{aligned}$$

This argument can be repeated for the other cases where negatives obligate us to flip the sign.

Answer (Total)

Take $\dfrac{a}{b}, \dfrac{c}{d} \in \mathbb{Q}$. Since $\leqslant_{\mathbb{Z}}$ is totally ordered we have

$$ad \leqslant_{\mathbb{Z}} bc \;\; \texttt{or} \;\; bc \leqslant_{\mathbb{Z}} ad$$

and thus by definition

$$\frac{a}{b} \leqslant_{\mathbb{Q}} \frac{c}{d} \;\; \texttt{or} \;\; \frac{c}{d} \leqslant_{\mathbb{Q}} \frac{a}{b}. \quad \square$$

Filler

Question

Suppose we have defined a `Matrix` class for handling matrixes. In order to implement all the arithmetic magic functions we would need to know the following:

1. What is the negation of a matrix? (What is the additive zero?)
2. What is the inversion of a matrix? (What is the multiplicative one?)

Not testable material ends.

Next Time

1. Review.
   1.1 Past exams.
   1.2 Your questions.