

CS 3331a - Assignment 1 - Solutions

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Question 1 - 20 marks

Show

$$\sum_{i=0}^n i = \frac{n(n+1)}{2} \tag{1}$$

Proof.

Base - 5 marks

Let $n = 0$ we have

$$\sum_{i=0}^0 i = 0 \text{ and } \frac{0 \cdot 1}{2} = 0$$

so (1) holds for $n = 0$.

Assumption - 5 marks

Assume that:

$$\sum_{i=0}^{n-1} i = \frac{(n-1)(n)}{2}$$

Induction - 10 marks

$$\begin{aligned} \sum_{i=0}^n i &= n + \sum_{i=0}^{n-1} i && \text{by def. of sum} \\ &= n + \frac{(n-1)(n)}{2} && \text{by assumption} \\ &= \frac{2n + n^2 - n}{2} \\ &= \frac{n(n+1)}{2} && \text{as desired} \end{aligned}$$

So (1) is proved by induction. □

Assuming $\sum_{i=0}^n i = \frac{(n)(n+1)}{2}$ and showing $\sum_{i=0}^{n+1} i = \frac{(n+1)(n+2)}{2}$ is fine as well.

Question 2 - 15 marks

Show $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$.

Proof. Suppose that $[a]_R \cap [b]_R \neq \emptyset$,

$$\begin{aligned}
 [a]_R \cap [b]_R \neq \emptyset &\Rightarrow \exists x, \text{ such that } x \in [a]_R \wedge x \in [b]_R && \text{def. of class} \\
 &\Rightarrow aRx \wedge bRx && \text{def. of class} \\
 &\Rightarrow aRb && \text{symmetry and transitivity} \\
 &\Rightarrow \forall c \in \Sigma, (aRc \Rightarrow bRc) && \text{symmetry and transitivity} \\
 &\Rightarrow \forall x \in [a]_R, (aRx \Rightarrow bRx \Rightarrow x \in [b]_R) && \text{def. of class} \\
 &\Rightarrow [a]_R \subseteq [b]_R
 \end{aligned}$$

By similar argument $[b]_R \subseteq [a]_R$ giving $[a]_R = [b]_R$. So it has been shown that either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$, as desired. \square

Marks were awarded for "reasonable progress"

Question 3 - 15 marks

Transitive closure

$$R^+ = R \cup \{(a, c), (a, d), (b, d), (c, c), (d, d)\}.$$

Reflexive transitive closure

$$R^* = R^+ \cup \{(a, a), (b, b), (e, e)\}$$

Deduct 2 marks for each missing element.

Question 4 - 15 marks

Let $L = \{aa, abc, cba\}$. The set of all sets of L , or the *power set* of L , denoted 2^L , is

$$\begin{aligned}
 2^L &= \{\emptyset\} \cup \{\{aa\}, \{abc\}, \{cba\}\} \cup \{\{aa, abc\}, \{aa, cba\}, \{abc, cba\}\} \cup \{\{aa, abc, cba\}\} \\
 &= \{\emptyset, \{aa\}, \{abc\}, \{cba\}, \{aa, abc\}, \{aa, cba\}, \{abc, cba\}, \{aa, abc, cba\}\}
 \end{aligned}$$

Deduct 2 marks for each missing element.

Question 5 - $3 \times 7 = 21$ marks

- The language $L_1 = \{a^n b^n | n \geq 0\}$ consists of all words starting with a sequence of a 's ending in a sequence of b 's of the same length. Since $x_1 = abab \neq a^n b^n$ for some n ($aabb$ would) x_1 is *not* in L_1 .
- The set $\{a, b\}^*$ contains *any* word one can construct with a, b , therefore $L_2 = \{waa | w \in \{a, b\}^*\}$ is the language consisting of all words ending with two a 's. It is easy to see that $x_2 = ababaa$ would be in this language.
- The language $L_3 = \{a^{2^n} | n \geq 0\}$ consists of sequences of a that are length some power of two. As $x_3 = aaaa = a^{2^2}$, x_3 is in the language.

Question 6 - 14 marks

The language $L_1 = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\}$ consists of all words of $\{a, b\}^*$ that have an equal number of a 's and b 's. The language $L_2 = \{a^i b^j \mid i, j \geq 0\}$ consists of all words of $\{a, b\}^*$ that begin with a sequence of a 's followed by a sequence of b 's. The words in both languages are those that begin with a sequence of a 's followed by an equally long sequence of b 's, or more simply

$$L = \{a^k b^k \mid k \geq 0\}.$$

TA Comments - Typical Mistakes

Question 1

Base case

1. If you assume that the base case is $n = 1$ then you must also show that the $n = 0$ case is true as this wouldn't be captured by the induction.
2. The left hand side and right hand side need to be worked out separately. Saying:

$$\sum_{i=0}^0 = \frac{0 \cdot 1}{0} = 0$$

is bad style. A *small* deduction was made for doing this. (Better now than on the midterm!)

3. You must **clearly state your assumption**.
4. Do not assume (1) for *some* n . It doesn't help to know that (1) works when say $n = 57$. What you must do is assume *any arbitrary* n satisfies (1) and then show this implies that $n + 1$ also satisfies (1).
5. The whole $LHS = RHS$ proof style is what we learned in high school, but is generally considered bad form. Dr. James Stewart (the guy who wrote your calculus textbook), once reamed me out in front of the class for doing a proof this way. I've never done it again.

Question 2

Proofs are hard and this question is not an exception to this rule. All I can suggest is that you avoid writing out your arguments in sentence form. It's easier to see / follow your argument when using the "language of math", i.e. symbols (that's why they were invented). Also, if at all possible, get someone to read your proof, if it doesn't make sense to them you've probably done something wrong.

If you feel that you deserve more marks for this question I would be glad to discuss it with you at my office hour. However I caution that you better have a strong argument as I spent a *lot* of time carefully combing through your proofs. Also, my last two degrees are in pure mathematics.

One nit-picky thing. The symbol for set inclusion is \in not ε .

Question 3

Nearly everyone (including me) forgot to include $(e, e) \in R^*$. Remember, the reflexive closure must satisfy xR^*x for every $x \in A = \{a, b, c, d, e\}$. I apologize to the first few people I marked and had to go back and correct. TA's make mistakes too I suppose!

The easiest way to do this question is by drawing the graph (on next page). Also, it's not cheating to check your work with someone else.

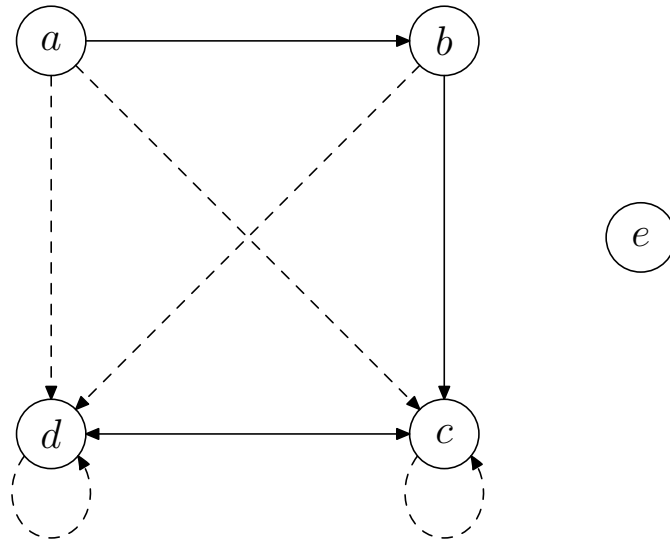


Figure 1: R^+ , the transitive closure, dashed arrows indicate the relations that are added. Note that the arrow connecting d and c is bidirectional.

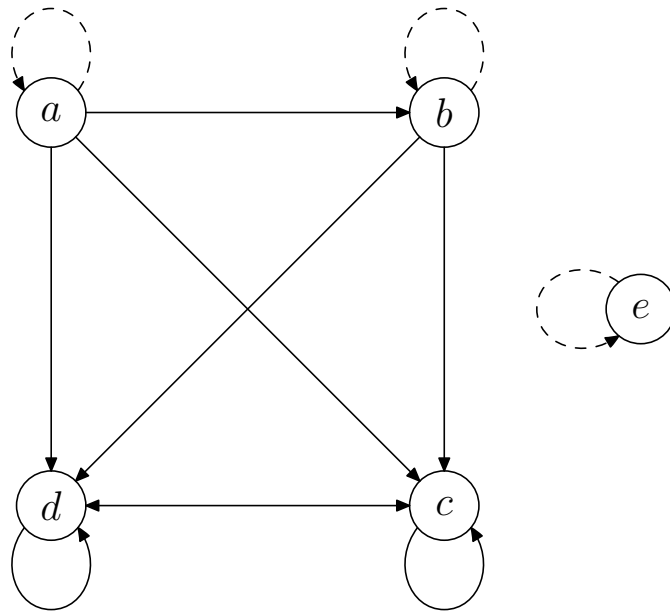


Figure 2: R^* , the reflexive transitive closure, dashed arrows indicate the relations that are added. Note that the arrow connecting d and c is bidirectional.

Question 4

The symbol $\emptyset = \{\}$ is the empty set, and is indeed in the power set of L . The symbol ε is the empty *word* and is **not** in the powerset. In fact ε isn't even a set. The set $\{\varepsilon\}$ isn't a subset of L either as $\varepsilon \notin L$.

Question 5

Read instructions carefully. Many did not include english descriptions or failed to address if $x_i \in L_i$.

Question 6

Saying $L = \{a^i b^i \mid i \geq 0\}$ got you full marks and is the most concise and preferable solution. Sometimes (especially in mathematics) less *is* more.