CS 3331a - Assignment 1 - Solutions

Paul Vrbik

September 28, 2011

Question 1 - 20 marks

Show

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

Proof.

Base - 5 marks

Let $n = 0$ we have

$$\sum_{i=0}^{0} i = 0 \text{ and } \frac{0 \cdot 1}{2} = 0$$

so (1) holds for $n = 0$.

Assumption - 5 marks

For $n > 0$ assume that:

$$\sum_{i=0}^{n-1} i = \frac{(n-1)(n)}{2}$$

Induction - 10 marks

$$\begin{aligned}
\sum_{i=0}^{n} i &= n + \sum_{i=0}^{n-1} i && \text{by def. of sum} \\
&= n + \frac{(n-1)(n)}{2} && \text{by assumption} \\
&= \frac{2n + n^2 - n}{2} \\
&= \frac{n(n+1)}{2} && \text{as desired}
\end{aligned}$$

So (1) is proved by induction. □

Assuming $\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$ *and showing* $\sum_{i=0}^{n+1} i = \frac{(n+1)(n+2)}{2}$ *is fine as well.*

Question 2 - 15 marks

Given an equivalence relation R over Σ and two arbitrary equivalence classes $[a]_R$ and $[b]_R$, where $a, b \in \Sigma$, prove that either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$.

Proof. Suppose that $[a]_R \cap [b]_R \neq \emptyset$,

$$
\begin{aligned}
[a]_R \cap [b]_R \neq \emptyset &\Rightarrow \exists x, \text{ such that } x \in [a]_R \wedge x \in [b]_R \\
&\Rightarrow aRx \wedge bRx && \text{def. of class} \\
&\Rightarrow aRb && \text{symmetry and transitivity} \\
&\Rightarrow \forall c \in \Sigma, (aRc \Rightarrow bRc) && \text{symmetry and transitivity} \\
&\Rightarrow \forall x \in [a]_R, (aRx \Rightarrow bRx \Rightarrow x \in [b]_R) && \text{def. of class} \\
&\Rightarrow [a]_R \subseteq [b]_R
\end{aligned}
$$

By similar argument $[b]_R \subseteq [a]_R$ giving $[a]_R = [b]_R$. So it has been shown that either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$, as desired. □

Marks were awarded for "reasonable progress"

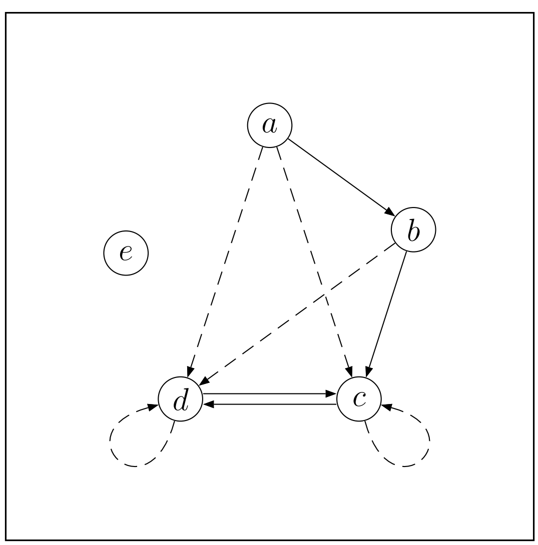
Question 3 - 15 marks

In the graphs to follow, dotted lines indicate the relations that were added.

Transitive closure

$$
\begin{aligned}
&R^1 = R \\
&R^2 = R^1 \circ R = \{(a, c), (b, d), (c, c), (d, d)\} \\
&R^3 = R^2 \circ R = \{(a, d), (c, d), (d, c), (b, c)\} \\
&R^4 = R^3 \circ R = \{(a, c), (d, c), (c, d), (b, d)\} && \text{no new pairs}
\end{aligned}
$$

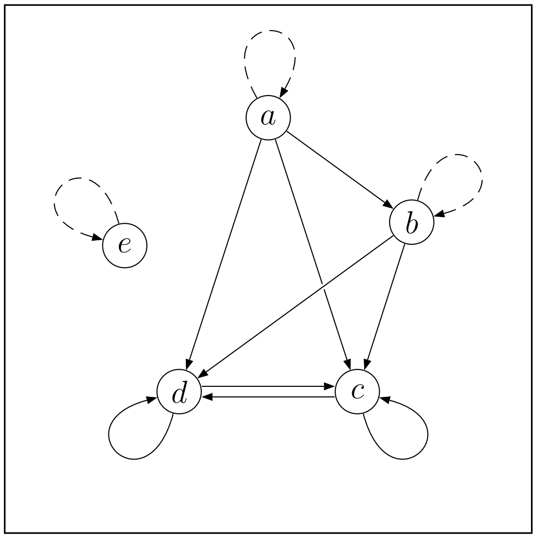
$$R^+ = R \cup \{(a, c), (a, d), (b, d), (c, c), (d, d)\}.$$



Reflexive transitive closure

$$R^0 = \{(a,a),(b,b),(e,e)\}$$

$$R^* = R^+ \cup \{(a,a),(b,b),(e,e)\}$$



Deduct 2 marks for each missing element. Graphs and explicit R^is were not required for full marks.

Question 4 - 15 marks

Let $L = \{aa, abc, cba\}$. The set of all sets of L, or the *power set* of L, denoted 2^L, is

$$\begin{aligned} 2^L &= \{\emptyset\} \cup \{\{aa\},\{abc\},\{cba\}\} \cup \{\{aa,abc\},\{aa,cba\},\{abc,cba\}\} \cup \{\{aa,abc,cba\}\} \\ &= \{\emptyset,\{aa\},\{abc\},\{cba\},\{aa,abc\},\{aa,cba\},\{abc,cba\},\{aa,abc,cba\}\} \end{aligned}$$

Deduct 2 marks for each missing element.

Question 5 - $3 \times 7 = 21$ marks

(a) The language $L_1 = \{a^n b^n | n \geq 0\}$ consists of all words starting with a sequence of a's ending in a sequence of b's such that the number of a's and b's, respectively, is equal. Since $x_1 = abab \neq a^n b^n$ for some n ($aabb$ would) x_1 is *not* in L_1.

(b) The set $\{a,b\}^*$ contains *any* word one can construct with a, b, therefore $L_2 = \{waa | w \in \{a,b\}^*\}$ is the language consisting of all words ending with two a's. It is easy to see that $x_2 = ababaa$ is in this language.

(c) The language $L_3 = \{a^{2^n} | n \geq 0\}$ consists of sequences of a's the length of which is some power of two. As $x_3 = aaaa = a^{2^2}$, x_3 is in the language.

Question 6 - 14 marks

The language $L_1 = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\}$ consists of all words of $\{a, b\}^*$ that have an equal number of a's and b's. The language $L_2 = \{a^i b^j | i, j \geq 0\}$ consists of all words of $\{a, b\}^*$ that begin with a sequence of a's followed by a sequence of b's. The words in both languages are those that begin with a sequence of a's followed by an equally long sequence of b's, or more simply

$$L = \{a^k b^k | k \geq 0\}.$$

TA Comments - Typical Mistakes

Question 1

Base case

1. If you assume that the base case is $n = 1$ then you must also show that the $n = 0$ case is true as this wouldn't be captured by the induction.

2. You must **clearly state your assumption**.

3. Do not assume (1) for *some* n. It doesn't help to know that (1) works when say $n = 57$. What you must do is assume *any arbitrary* n satisfies (1) and then show this implies that $n+1$ also satisfies (1).

Question 2

Proofs are hard and this question is not an exception to this rule. If at all possible, get someone to read your proof, if it doesn't make sense to them you've probably done something wrong.

One nit-picky thing. The symbol for set inclusion is $\in$ not ε.

Question 3

Do not forgot to include $(e, e) \in R^*$. Remember, the reflexive closure must satisfy xR^*x for every $x \in A = \{a, b, c, d, e\}$.

The easiest way to do this question is by drawing the graph. Also, it's not cheating to check your work with someone else.

Question 4

The symbol $\emptyset = \{\}$ is the empty set, and is indeed in the power set of L. The symbol ε is the empty *word* and is **not** in the powerset. In fact ε isn't even a set. The set $\{\varepsilon\}$ isn't a subset of L either as $\varepsilon \notin L$.

Question 5

Read instructions carefully. Many did not include english descriptions or failed to address if $x_i \in L_i$.

Question 6

Saying $L = \{a^i b^i | i \geq 0\}$ got you full marks and is the most concise and preferable solution. Sometimes (especially in mathematics) less *is* more.

Typeset by: Paul Vrbik Last Compiled: September 28, 2011