CS 3331a - Assignment 1 - Solutions

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Question 1 - 21 marks

Given an equivalence relation R over Σ and two arbitrary equivalence classes $[a]_R$ and $[b]_R$, where $a, b \in \Sigma$, prove that either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$.

Proof. Suppose that $[a]_R \cap [b]_R \neq \emptyset$,

$$\begin{aligned}
[a]_R \cap [b]_R \neq \emptyset &\Rightarrow \exists x,\ \text{such that } x \in [a]_R \wedge x \in [b]_R && \\
&\Rightarrow aRx \wedge bRx && \text{def. of class} \\
&\Rightarrow aRb && \text{symmetry and transitivity} \\
&\Rightarrow \forall c \in \Sigma, (aRc \Rightarrow bRc) && \text{symmetry and transitivity} \\
&\Rightarrow \forall x \in [a]_R, (aRx \Rightarrow bRx \Rightarrow x \in [b]_R) && \text{def. of class} \\
&\Rightarrow [a]_R \subseteq [b]_R &&
\end{aligned}$$

By similar argument $[b]_R \subseteq [a]_R$ giving $[a]_R = [b]_R$. So it has been shown that either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$, as desired. □

Marks were awarded for "reasonable progress"

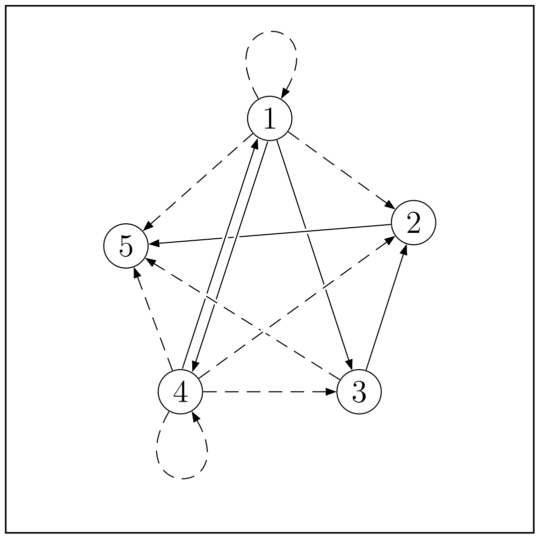
Question 2 - 20 marks

Let $A = \{1, 2, 3, 4, 5\}$ be a set and $R = \{(1, 3), (1, 4), (2, 5), (3, 2), (4, 1)\}$ a relation defined on $A \times A$. Find the transitive closure and the reflexive and transitive closure.

Solution. In the graphs to follow, dotted lines indicate the relations that were added.

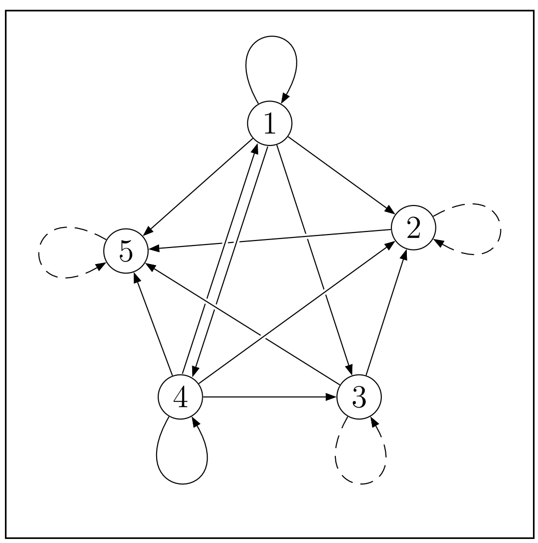
Transitive closure

$$R^+ = R \cup \{(1, 1), (1, 2), (1, 5), (3, 5), (4, 2), (4, 3), (4, 4), (4, 5)\}.$$



Reflexive closure

$$R^* = R^+ \cup \{(2, 2), (3, 3), (5, 5)\}.$$



Graphs were not required for full marks. ■

Question 3 - 20 marks

Prove by induction on n that

$$1 + 3 + \cdots + (2n - 1) = n^2 \tag{1}$$

Solution.

Base - marks

Let $n = 1$ we have

$$\text{LHS} = 1 \text{ and } \text{RHS} = n^2\Big|_{n=1} = 1^2 = 1$$

so (1) holds for $n = 1$.

Assumption - marks

Assume that:

$$1 + 3 + \cdots + (2n - 1) = n^2$$

Induction - marks *Show that the assumption implies (1) at $n + 1$.*

$$\begin{aligned} 1 + 3 + \cdots + (2n-1) + (2(n+1) - 1) &= n^2 + (2(n+1) - 1) && \text{by assumption} \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 && \text{as desired} \end{aligned}$$

So (1) is proved by induction. ■

Question 4 - 18 marks

Give a brief English descriptions for each of the following languages and answer whether each of the given words is in the language:

(a) $L_1 = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\}$. $x_1 = abbaab$, $x_2 = baababbba$.

Solution. L_1 consists of all words constructible with the letters a and b such that the number of as and bs in the word are equal. Trivially, $x_1 \in L_1$ and $x_2 \notin L_1$. ■

(b) If x is in L_2, then axb and bxa are in L_2; if x and y are in L_2, then xy is in L_2; and no other words are in L_2. $y_1 = bbbaaa$, $y_2 = baaababb$.

Solution. Is again the language consisting of words with an equal amount of as and bs.

Trivially $y_1 \in L_2$ as it can be generated by repeated application of the rule $bxa \in L_2$. Also, if we write

$$y_2 = (ba)(a(a(ba)b)b)$$

we see that $y_2 \in L_2$ as well.

■

(c) $L_3 = \{w \in \{0,1\}^+ \mid \text{the value of } w \text{ as a binary number is divisible by} 5\}$. $z_1 = 01010$, $z_2 = 1101$.

Solution. L_3 is the set of binary numbers (allowing arbitrary leading zeros) that are divisible by 5 (base ten). z_1, which is 10 base ten, and therefore divisible by 5 *is* in the language. z_2, which is 13 base ten, is not. ■

Question 5 - 21 marks

Prove that $(AB)^R = B^R A^R$ for arbitrary given languages A and B.

Proof. Let $x \in (AB)^R$ (and show $x \in B^R A^R$).

Since $x \in (AB)^R$ we can write

$$x = (uv)^R$$

with $u \in A$ and $v \in B$ (by definition of reversal of languages).

Letting $u = u_0 \cdots u_n$ and $v = v_0 \cdots v_m$ where $u_i, v_j \in \Sigma$ we can write

$$\begin{aligned} x &= (u_0 \cdots u_n \cdot u_0 \cdots u_m) & \\ &= (v_m \cdots v_0 \cdot v_n \cdots v_0) & \text{by definition of reversal} \\ &= v^R u^R & \text{by definition of reversal} \end{aligned}$$

Since $v^R \in B^R$ and $u^R \in A^R$ (by definition) we have

$$\begin{aligned} v^R u^R \in B^R A^R &\Rightarrow x \in B^R A^R \\ &\Rightarrow \left(x \in (AB)^R \Rightarrow x \in B^R A^R\right) \\ &\Rightarrow (AB)^R \subseteq B^R A^R \end{aligned}$$

This same argument in reverse yields $B^R A^R \subseteq (AB)^R$ which gives

$$(AB)^R = B^R A^R$$

as desired.

□